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Implied Volatility Surfaces in the South African ALSI Market

by

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Report presented in partial fulfilment
of the requirements for the degree of
BCommHons (Financial Risk Management)
at the University of Stellenbosch

Supervisor: Mr C.J. van der Merwe

Degree of confidentiality: A

October 2016

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Acknowledgements

The author would like to thank the following people for their contribution towards this project:

- My supervisor, Mr. Carel van der Merwe, for his guidance, patience, time and support during the duration of this research assignment.
- Prof. Conradie for the opportunity to further my studies in a post graduate program.
- My parents, Manie and Alice Cronjé, thank you for your unconditional love and support. Thank you for always believing in me and giving me the opportunity to study in the first place.
- Christo and Elzaan Cronjé for the support and love during this research process.
- Jurie Taljaard for all his contributions towards proofreading.
- Karla Taljaard for the immense encouragement, support and love. Thank you for always helping me move forward.
- Lastly, to our Lord Jesus Christ in Heaven, who without none of this would be possible.

Abstract

The constant volatility assumption in the Black-Scholes model has led to the development of alternative models. These models capture the effect of the implied volatility from the market, with respect to the variation in time. That is, volatility is volatile in itself, which can lead to substantial errors in estimation of option prices. In illiquid markets it is difficult to obtain volatility surfaces due to the lack of data. Volatility surfaces enables investors to price options for any set of strike prices and time to maturity of the option. In this research assignment two different models are implemented in the South African market context. The first model is the Stochastic Alpha Beta Rho (SABR) model developed by Hagen et al. (2002) and adjusted by West (2005) to fitted illiquid markets. The second model is a deterministic approach, where a quadratic function is fit to the market data, developed by Kotzé et al. (2009). The SABR model initially did not fit the market data, but the author of this paper proposed an adjustment to the SABR volatilities to obtain a reasonable fit. This research paper found that the Quadratic Deterministic model results in more accurate results and it is easier to implement when compared to the SABR model.

Key words:

Volvol, SABR, ATM, Quadratic Deterministic, Options, Black-Scholes, Volatility Surface, ALSI, SAFEX, JSE, Moneyness

Opsomming

Die konstante volatiliteit-aanname in die Black-Scholes model het gelei tot die ontwikkeling van alternatiewe modelle. Hierdie alternatiewe modelle sluit in hoe volatiliteit, geïmpliseer deur die mark, varieer met betrekking tot tyd. Dit beteken dat volatiliteit self volatiel is, wat tot aansienlike foute kan lei in die beraming van opsie-pryse. Dit is moeilik om volatiliteit- oppervlaktes te kry in illikiede markte, as gevolg van die tekort aan markdata. ‘n Volatiliteitsoppervlakte stel die belegger in staat om opsies te prys van enige stel trefpryse en tyd tot die vervaldatum van die opsie. Hierdie navorsingsprojek identifiseer en implementeer twee modelle in die illikiede Suid-Afrikaanse mark. Die eerste model is die Stogastiese Alpha Beta Rho (SABR)-model wat deur Hagen et al. (2002) ontwikkel is en deur West (2005) aangepas is om ‘n illikiede mark te pas. Die tweede model is ‘n deterministiese model wat ontwikkel is deur Kotzé et al. (2009). Die SABR-model het aanvanklik nie die markdata goed gepas nie. Hierdie navorsingsprojek het ‘n eenvoudige manier voorgestel en geïmplementeer wat gesorg het dat die model redelik goed pas. Die skrywer van hierdie navorsingsprojek het gevind dat die Kwadratiese Deterministiese-model meer akkurate resultate oplewer. Dit is ook makliker om te implementer wat vir kleiner maatskappye, met moontlike beperkte hulpbronne, die beter opsie sal wees.

Sleutelwoorde:

Volvol, SABR, ATM, Kwadraties Deterministies, Opsies, Black-Scholes, Volatiliteitsoppervlakte, ALSI, SAFEX, JSE

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List of abbreviations and/or acronyms

ATM	At-the-money
ALSI	All Share Index
DVF	Deterministic Volatility Functions
FTSE	Financial Times Stock Exchange
JSE	Johannesburg Stock Exchange
MSE	Mean Square Error
Mat	Maturity
OTC	Over-the-counter
SAFEX	South African Futures Exchange
Vol	Volatility
volvol	Volatility of Volatility
S&P	Standard & Poor
Mib	Milano Italia Borsa
SABR	Stochastic Alpha Beta Rho
CIR	Cox-Ingersoll-Ross

CHAPTER 1
INTRODUCTION

1.1 INTRODUCTION

Sircar and Papanicolaou (1998) stated that there is an immense pressure on practitioners to change to the risk-neutral world as demonstrated by the use of the implied Black-Scholes volatility measure on market prices. Volatility is assumed constant in the Black-Scholes model, which is a problem in a real world scenario where volatility itself varies over time. This means that using the Black-Scholes model can lead to errors when estimating implied volatility surfaces. The SABR (Stochastic Alpha Beta Rho) model was developed by Patrick Hagan, Deep Kumar, Andrew Lesniewski and Diana Woodward to account for the problem where volatility is assumed constant. The SABR model allows the volatility itself to be a function of time, thus volatility can change as time changes.

In liquid markets, the SABR model is capable of approximating volatility smiles regardless of the estimation method used (Hansen, 2011:81). The problem with the standard SABR model is that in an illiquid market, like the South African Equity Derivatives Market and Over-the-counter (OTC) market, there is a shortage of input data to calibrate the volatility smiles. The Equity Derivatives Market of South Africa was formerly known as the South African Futures Exchange (SAFEX).[1]The shortage arises due to traders not publishing the input rates needed to estimate volatility smiles. Only banks have access to the OTC input data. West (2005) made certain adjustments to the standard SABR model and developed an algorithm which is robust and could possibly be used for mark to market and hedging of option portfolios. This means that the implied volatility surfaces in an illiquid market can be obtained with the possibility of minimal model errors.

Kotzé and Joseph (2009) took a more deterministic approach. They showed how to generate the implied volatility surface using a quadratic deterministic function fitted to implied volatility data from the All Share Index (ALSI) options traded on the Equity Derivatives Market. Their research further showed that the deterministic function gives an implied volatility shape, which seems stable over time.

West's adjusted SABR model is relatively more complex and difficult to implement compared to the simpler quadratic deterministic approach taken by Kotzé and Joseph. In this research assignment the two approaches will be compared to find out which approach gives the most

[1] The JSE acquired SAFEX in 2001, and is now called the Equity Derivatives Market.

accurate outputs, taking into consideration the complexity of the adjusted SABR model and simplicity of the deterministic approach, which can have an impact on efficiency.

1.2 PROBLEM STATEMENT

In illiquid markets, there might not be enough data to calibrate different volatility models. The SABR volatility model uses the at-the-money (ATM) volatility as an input. Kotzé and Joseph (2009:3) identified this as a problem in illiquid markets, because the market surfaces generated by this model will have errors due to a shortage in ATM traded options in the market. There are not enough ATM traded options in the South African market to generate enough data to calibrate the model.

The second problem with the SABR model is that the parameters of the model (alpha, beta and rho) are independent of time (parameters are time-homogenous). Kotzé (2011:117) explained this to be a problem, because the volatility surfaces in the future will be the same as the volatility surfaces generated today. This means that the parameters must be estimated on a regular basis, which might be a time consuming process.

1.3 RESEARCH QUESTION

Which model results in more accurate estimations of the implied volatility surface in an illiquid market: estimating the volatility surface with a quadratic deterministic approach, or using the West’s adjusted SABR model?

1.4 RESEARCH OBJECTIVES

The main objective of this research assignment is to be able to investigate which prescribed method yields minimal errors to generate volatility surfaces in illiquid markets. This research assignment will give an in depth explanation of how the SABR model works and how it has been adjusted by West (2005) to fit an illiquid market. Furthermore, the simpler deterministic approach can be compared with the complexity of the adjusted SABR model to determine the trade-off between accuracy and efficiency.

In South Africa the index options are traded on the ALSI futures contracts which is listed on the Equity Derivatives Market. This will be the source of the raw data needs for this study. The parameters will be estimated by minimising the sum of square errors of the observed option prices.

1.5 BENEFITS OF THE STUDY

This research assignment would be able to add to the reasonableness of certain models developed to capture the implied volatility smiles in an illiquid market. De Araujo and Mare

(2006:15) stated that illiquid traded options prove to be problematic in the South African equity derivatives market. This study would be able to give an indication of which approach is better to generate volatility smiles from an illiquid market. It will be of importance to entities with limited resources and time to test the reasonableness of the different approaches, as estimating volatilities can be relatively more expensive for smaller firms. The benefits of testing the two approaches in the South African market will help to further explain and understand the outputs and estimations of the different models. This study will also investigate the different ways in which banks estimate volatilities in the South African market.

1.6 CHAPTER OUTLINE

This section provides the reader with a broad overview of the content in this research assignment. The chapter following immediately after this is the literature review chapter in which previous research are set out and discussed. This is done to understand where the two models in question arise from. Chapter 2 is divided into three sub-chapters. In Chapter 2.2 the Black-Scholes model is briefly explained to get an understanding of how constant volatility is included into the model and why this assumption leads to the use of alternative models in estimating option prices. The SABR model developed by Hagen et al. (2002) is reviewed in Chapter 2.3. This is the first model in question. The model dynamics is discussed followed by the mathematical procedures in estimating the model parameters. Chapter 2.4 sets out the previous research that leads up to the Quadratic Deterministic model developed by Kotzé et al. (2009) and concludes by reviewing the theory that fully describes the volatility surface.

Chapter 3 deals as the research methodology chapter in which the methods to implement the two models are discussed. This chapter is split up into three parts. In Chapter 3.2 a description of the data used is given, that is a description of the ALSI input data used. The second part of this chapter, Chapter 3.3, presents the methodology used to implement the SABR model in the South African context by using the defined formulas from Chapter 2.3. Following the SABR model is the methodology concerning the Quadratic Deterministic approach to estimate implied volatility surfaces. Chapter 3.4 sets out the methods and functions to implement this model. The chapter concludes by presenting a description of the comparison methods to determine how well the models fit the ALSI market data.

After the methodology to implement the models follows the results and findings from implementing the two models, which is found in Chapter 4. In Chapter 4.2 the volatility skews for different expiries are shown to express the relationship between the volatilities implied from the market and the corresponding moneyness.[2] Chapter 4.3 sets out the results obtained from the SABR model and Chapter 4.4 the results obtained from the Quadratic Deterministic model.

The results are summarised in Chapter 5, together with an overall conclusion and recommendations for further research on implied volatility surfaces in the South African market. This chapter also compares the SABR model and Quadratic Deterministic approach to give the reader an informative indication on which model captures the essence of *volatility of volatility* the best.

[2] $Moneyness = \frac{Strike}{Spot}$

CHAPTER 2
LITERATURE REVIEW

2.1 INTRODUCTION

In this chapter, literature regarding implied volatility surfaces is discussed. Cont and Da Fonseca (2002:45) wrote that implied volatility surfaces of option prices changes dynamically over time, which is not taken into account by conventional modelling approaches like the model developed by Black (1976). The sub section that follows defines the Black-Scholes-Merton Model and pricing formulas to enable the reader to understand what assumptions led to the development of alternative models. Two alternative models namely the SABR model and Quadratic Deterministic approach are discussed following the Black-Scholes-Merton model.

2.2 THE BLACK-SCHOLES-MERTON PRICING FORMULAS

The following section serves as an introduction to fully understand the mathematics underlying the two models in question. The Black-Scholes method is widely used in the financial world and is therefore important to first understand what it entails and why alternative models are needed to capture the essence of volatility.

Hull (2012) defines the Black-Scholes-Merton formulas for the price of European call and put options as the following:

$$c = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$$

and

$$p = K e^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1)$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

S_0 is defined as the stock price at time zero, K the strike, r the risk-free interest rate, q the dividend yield and T the time to maturity. The function $N(x)$ is the cumulative probability function of the standardised normal distribution. This means, if x has a standard normal distribution, the probability that the random variable is less than x.

The Black-Scholes-Merton formulas can be derived by solving the Black-Scholes-Merton differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

Subject to the boundary conditions for a European call option $f = \max(S - K, 0)$, when $t = T$, and European put option $f = \max(K - S, 0)$, when $t = T$.

The following assumptions are made in the derivation of the above differential equation

- The stock price follows the following process with μ and σ constant

$$\frac{\mathrm{d}S}{S} = \mu \mathrm{d}t + \sigma \mathrm{d}z \tag{2.2.1}$$

  with $\mathrm{d}z$ defined as the basic Wiener process.
- Short selling of securities is allowed.
- There are no taxes or transaction costs in the buying and selling of the stock.
- No dividends are paid during the derivative's life.
- No arbitrage opportunities.
- Continuous security trading.
- The risk-free rate is the same for all maturities and constant.
- The option is "European", which means it can only be exercised at maturity of the option.

Black (1976) made adjustments to the original Black-Scholes-Merton model in order to price interest rate derivatives like options on futures. This is done in order to avoid estimating the income from the underlying asset, which is already included into the future's price. Hull (2012:372) states that the future price incorporates the market's estimate of its income. The Black-76 model discounts a forward rate F, instead of using the stock price S_0 in the pricing formulas. Therefore, substitute S_0 equal to $Fe^{-(r-q)T}$ into the original Black-Scholes-Merton pricing formulas as follows

$$c_{76} = e^{-rT}[FN(d_1) - KN(d_2)] \tag{2.2.2}$$

$$p_{76} = e^{-rT}[KN(-d_2) - FN(-d_1)] \tag{2.2.3}$$

where

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{F}{K}\right) - \left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

and σ is the volatility of the futures price, F.

The assumption that volatility in the Black-Scholes model is constant is the reason for alternative models being developed and used instead of the Black-Scholes model. Volatility is not constant which resulted in models being developed to capture this so called volatility of volatility or "volvol". In the following sections the SABR- and Quadratic Deterministic models are defined and explained to fully understand how each of them captures the volvol.

2.3 THE SABR MODEL

Hagen et al. (2002) derived the Stochastic Alpha Beta Rho Model where the forward value and volatility follows, under the forward measure, the processes

$$\mathrm{d}\hat{F} = \hat{\alpha}\hat{F}^{\beta}\mathrm{d}W_1, \qquad \hat{F}(0) = f$$

$$\mathrm{d}\hat{\alpha} = \nu\hat{\alpha}\mathrm{d}W_2, \qquad \hat{\alpha}(0) = \alpha$$

The forward price and volatility processes are correlated by

$$\mathrm{d}W_1\mathrm{d}W_2 = \rho dt$$

with ρ the correlation coefficient and W_1 and W_2 two Wiener processes. In the above equations β is the skewness parameter subject to the constraint $0 \leq \beta \leq 1$, and $\hat{\alpha}$ is the volatility. Furthermore, ν is defined as the volatility of volatility or volvol parameterwhich satisfies $\nu \geq 0$.

These processes show that the forward price, F, as well as the volatility, α, is stochastic. The assumption of constant volatility in Black's model can thus be dropped with the SABR model. formula (2.2.1) show how the volatility in Black's model is constant and only depends on time, where the SABR includes extra shocks or randomness by adding a stochastic term to the process. This extra volatility is defined in the $\mathrm{d}\hat{\alpha}$ term. To scale this extra randomness, the volvol parameter ν is included. Hagen et al. (2002) found that the SABR model has the tendency to be the simplest model which is homogenous in F and α.

The prices of European call and put options for the SABR model can be obtained from Black-76's formulas. Refer to formula (2.2.2) and formula (2.2.3), which is the Black-Scholes call- and put option prices, respectively. The implied volatility, $\sigma_B(K, f)$, is then obtained from these prices. In Hagen et al. (2002) the approximation to the implied volatility is given by

$$\sigma_B(K,f) = \frac{\alpha}{(fK)^{\frac{1-\beta}{2}}\left\{1 + \left[\frac{(1-\beta)^2}{24} ln\left(\frac{f}{K}\right) + \frac{(1-\beta)^4}{1920} ln\left(\frac{f}{K}\right) + o(K,f)\right]\right\}} \times \left(\frac{z}{x(z)}\right) \times$$

$$\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\alpha\nu}{(fK)^{\frac{1-\beta}{2}}} + \frac{2-3\rho^2}{24}\nu^2\right]T + o(K,f)\right\} \quad (2.3.1)[3]$$

where $z = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\ln(\frac{f}{K})$

and $x(z) = \ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right)$

When the option is at-the-money, the strike price equals the spot price. The moneyness, $\frac{Strike}{Spot} = \frac{K}{f}$, of an option is then equal to one, thus $\ln\left(\frac{K}{f}\right)$ equals zero. This leads to a reduction of formula (2.3.1), which simplifies the calculation of the ATM volatilities implied by the SABR model. The formula then simplifies to

$$\sigma_{ATM} = \sigma_B(f,f) = \frac{\alpha}{f^{(1-\beta)}}\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2-2\beta}} + \frac{1}{4}\frac{\rho\beta\alpha\nu}{f^{(1-\beta)}} + \frac{2-3\rho^2}{24}\nu^2\right]T + o(K,f)\right\} \quad (2.3.2)$$

In formula (2.3.2), when $\beta = 1$ the model represents a stochastic log normal model, which means that the stochastic process takes on the form

$$\mathrm{d}\widehat{F_t} = \widehat{\alpha_t}\widehat{F_t}\mathrm{d}W_t$$

When $\beta = 0$ the process represents a stochastic normal model of the form

$$\mathrm{d}\widehat{F_t} = \widehat{\alpha_t}\mathrm{d}W_t$$

Chapter 2.3.2 further elaborates on these two specific cases for β.

The following sub-chapter sets out the dynamics of the SABR model. This is done to understand the qualitative behaviour of the model. The following approximations should only be used for demonstration purposes and not for pricing of an option.

2.3.1 Model Dynamics

Hagen et al. (2002) developed the following approximation for $\sigma_B(K,f)$ to study the qualitative behaviour of the SABR model

[3] $o(K,f)$ simply means that the following terms tend to zero as the expansion gets larger

$$\sigma_B(K,f) = \frac{\alpha}{f^{(1-\beta)}}\left\{1 - \frac{1}{2}(1-\beta-\rho\lambda)\ln\left(\frac{K}{f}\right) + \frac{1}{12}[(1-\beta)^2 + (2-3\rho^2)\lambda^2]\left(ln\left(\frac{K}{f}\right)\right)^2 + o(K,f)\right\} \quad (2.3.3)$$

with the strike K not too far away from the current forward f.

The local volatility is the implied volatility when the option is ATM, thus when the strike price is equal to the spot price. In formula (2.3.3) the local volatility is therefore the first factor defined as $\frac{\alpha}{f^{1-\beta}}$. Compared to the local volatility, $\lambda$is defined as

$$\lambda = \frac{\nu}{\alpha} f^{1-\beta}$$

which measures the strength ν of the volatility of the volatility ("volvol"). Hagen et al. (2002:90) stated that this formula should not be used to price real deals, it is only accurate enough to demonstrate the qualitative behaviour of the SABR model.

The *smile* and *skew* refer to the implied volatility as a function of f and K. The *backbone* refers to the curve generated by ATM volatility plots with a variation in f during normal trading. Thus for ATM options the backbone is essentially $\sigma_B(f,f) = \frac{\alpha}{f^{1-\beta}}$, which is the first term in equation (2.3.2). From this Hagen et al. (2002:90) observed, for $\beta = 0$, the backbone to be downward sloping and $\beta = 1$ results in a nearly flat backbone. This means the backbone is dependent on the value of β.

The second term in equation (2.3.2) represents the slope of the implied volatility with respect to the strike K. Hagen et al. (2002) refers to the second term as the skew and views it as a summation of two terms known as the *beta skew* and the *vanna skew.* Thus,

$$-\frac{1}{2}(1-\beta)\ln\left(\frac{f}{K}\right)$$

represents the beta skew. Since $0 \leq \beta \leq 1$, the beta skew is downward sloping. This results from the local volatility being a decreasing function of the forward price.

The vanna skew,

$$\frac{1}{2}\rho\lambda \ln\left(\frac{f}{K}\right)$$

is the skew caused by the correlation between the asset price and volatility. The negative correlation ($\rho < 0$) between the price of an asset and its volatility causes the vanna skew to be downward sloping on average.

2.3.2 SABR parameter estimation

Tran and Weigardh (2014:39) described the calibration of the SABR model as minimising the gap between the observed and predicted implied volatilities fitted by the SABR model for each strike. The mathematical term for this process is called the Least Square Method and is defined as

$$\min_{\nu,\alpha_0,\rho,\beta} \sum_i (\bar{\sigma}_i - \sigma_B(\nu,\alpha_0,\rho,\beta;K_i,f))^2, \qquad (2.3.2.1)$$

where $\bar{\sigma}_i$ denotes the i[th] market observed implied volatility and $\sigma_B(\nu,\alpha_0,\rho,\beta;K_i,f)$ denotes the implied volatilities from the SABR model subject to the SABR parameters given the K_i^{th} strike and ATM forward price f.

2.3.2.1 Estimating β

According to Hagen et al. (2002), the estimation of β can be determined in two ways. First from historical observations and secondly selected from "aesthetic considerations".

The historical observations method is approached by first taking the natural logarithm in equation (2.3.2).

$$ln\ \sigma_{ATM} = ln\ \sigma_B(f,f) = ln\ \alpha - (1-\beta) ln\ f\ +$$

$$ln\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2-2\beta}} + \frac{1}{4}\frac{\rho\beta\alpha\nu}{f^{(1-\beta)}} + \frac{2-3\rho^2}{24}\nu^2\right]T + o(f)\right\} \qquad (2.3.2.2)$$

This equation can be approximated by

$$ln\ \sigma_{ATM} \approx ln\ \alpha - (1-\beta) ln\ f$$

The third term in equation (2.3.2.2) is generally less than one or two percent and can therefore be ignored in estimating β. The parameter is extracted from log-log plots of historical observations of pairs of f and σ_{ATM}. This is done by plotting the natural logarithm of the ATM volatility on the y-axis and the natural logarithm of the forward price on the x-axis. West (2009:7) suggested that a time weighted regression is preferred in extracting an estimate for β. Hagen et al. (2002:91) found the historical observation approach to become quite noisy from both f and α being stochastic.

An *aesthetic consideration* is a way of selecting the parameter due to the user's certain believes about the data to be analysed. Three pre-determined estimations for β will be discussed, that is $\beta \in \{0, 0.5, 1\}$. This results in three common models as shown by Hansen (2011:43):

- **$\boldsymbol{\beta = 0}$ - Stochastic Normal Model**

  Setting the exponent $\beta = 0$ results in the forward process with the form

  $$\mathrm{d}f_t = \alpha_t \mathrm{d}W_t$$

  This forward process is stochastic normally distributed in the sense that its increments are stochastic normally distributed with a mean of zero and log normal standard deviation. This model is not usually suited for practical purposes as setting $\beta = 0$ will enable the forward process to be negative (Hansen, 2011:43).

- **$\boldsymbol{\beta = 0.5}$ - Stochastic CIR model**

  The Stochastic Cox, Ingersoll and Ross model, or more commonly known as the CIR model, is derived from the interest rate equilibrium model of Cox et al. (1985) and takes on the form

  $$\mathrm{d}f_t = \alpha_t f_t^{1/2} \mathrm{d}W_t$$

  The stochastic CIR model prevents the process from becoming negative in the "$f_t^{1/2}$"-term. This non-negative feature gives the CIR model an advantage to the Normal stochastic model mentioned above and is thus the preferred model.

- **$\boldsymbol{\beta = 1}$ - Stochastic Lognormal Model**

  The forward process now becomes

  $$\mathrm{d}f_t = \alpha_t f_t \mathrm{d}W_t$$

  which is similar to the standard geometric Brownian motion in the Black-Scholes model with the difference of the SABR model's volatility is a volatility itself, compared to the assumption in the Black-Scholes model that the volatility is constant. Setting $\beta = 1$ implies the forward rates being log normally distributed. This means the stochastic lognormal model contains the same property, as the CIR model, of non-negativity.

2.3.2.2 Estimating α, ν and ρ

There are two methods of estimating these parameters. The first method was developed by Hagen et al. (2002). Tran and Weigardh (2014:41) set out the following steps, which is used in estimating the parameters for the Hagen et al. (2002) method. First starting with the ATM volatility function stated above,

$$\sigma_{ATM} = \sigma_B(f,f) = \frac{\alpha_0}{f^{(1-\beta)}}\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha_0^2}{f^{2-2\beta}} + \frac{1}{4}\frac{\rho\beta\alpha_0\nu}{f^{(1-\beta)}} + \frac{2-3\rho^2}{24}\nu^2\right]T\right\}$$

$$\Leftrightarrow 0 = A\alpha_0^3 + B\alpha_0^2 + C\alpha_0 - \sigma_{ATM}f^{(1-\beta)} \qquad (2.3.2.3)$$

where

$$A = \left[\frac{(1-\beta)^2T}{24f^{(2-2\beta)}}\right], B = \left[\frac{\rho\beta\nu T}{4f^{(1-\beta)}}\right] and\ C = \left[1 + \frac{2-3\rho^2}{24}\nu^2T\right].$$

The steps in estimating α, ρ and ν are:

1. Choose initial values for ρ and ν .
2. By using the input values for β, ρ and ν, solve α_0 with equation (2.3.2.3).
3. Insert all parameters back into equation (2.3.3) to calculate σ_B for every strike.
4. Minimise the objective function

$$\min_{\nu,\rho}\sum_i[\bar{\sigma}_i - \sigma_B(\nu, \alpha_0(\nu, \rho, \sigma_{ATM}), \beta; K_i, f)]^2$$

   to get a new set of parameter values for ρ and ν.

5. Repeat steps 2 and 3 to get a new set of parameters and σ_B.
6. Plug the new σ_B back into the given objective function, then compare the value with a certain convergence criteria.
7. Move on to the next iteration until the convergence is met by a level of tolerance.

West (2005:7) suggested that it is possible for the cubic to have more than one real root, in which case you should choose the smallest positive root. This study will assume the West (2005) approach.

The second method entails using common optimisation techniques such as the Newton-Raphson Method for finding real roots or minimising the sum of squared errors to solve the minimisation problem.[4]

[4]Although the second method is mentioned, its application is beyond the scope of this text.

2.4 QUADRATIC DETERMINISTIC APPROACH

Kotzé and Joseph (2009) suggested fitting a quadratic function to the market data in order to obtain implied volatilities from the same deterministic model. This section first sets out different deterministic models before reviewing the quadratic function used for the ALSI volatility surface.

2.4.1 Deterministic Models

Deterministic volatility functions (DVF) are volatility models requiring no assumptions about the dynamics of the underlying process that generates the volatility (Kotzé et al. 2009:10). When the local volatility rate is a deterministic function of the asset price and time, the partial differential equation for option price dynamics is the well-known Black Scholes (1973) equation,

$$-\frac{1}{2}\sigma^2(F,t)F^2\frac{\partial^2 c}{\partial F^2} = \frac{\partial c}{\partial t}, \qquad (2.4.1.1)$$

where F is the forward price, c the forward option price, t the current time and $\sigma(F,t)$ the local volatility of the price. Dumas, Fleming and Whaley (1998:2068) rewrote equation (2.4.1.1) to be applicable for future and forward contracts, with initial condition $c(K,0) = max(S-K,0)$, as

$$\frac{1}{2}\sigma^2(F,T)K^2\frac{\partial^2 c}{\partial K^2} = \frac{\partial c}{\partial t}, \qquad (2.4.1.2)$$

In equation (2.4.1.2), K is defined as the strike price, T the time to expiration and $c(K,T)$ the call option price. Now valuing European-style options with the same time to expiration can be done simultaneously.[5] Dumas et al. (1998:2068) suggested the following deterministic functions for the implied volatility, as they mention it to be an arbitrary function,

Model 0: $\sigma = a_0$

Model 1: $\sigma(K) = a_0 + a_1K + a_2K^2$

Model 2: $\sigma(K,T) = a_0 + a_1K + a_2K^2 + a_3T + a_5KT$

Model 3: $\sigma(K,T) = a_0 + a_1K + a_2K^2 + a_3T + a_4T^2 + a_5KT$

The variables a_i, $i = 0,\ldots,5$, are determined by fitting the deterministic functions to the traded option data. Model 0 is the Black-Scholes model with a constant volatility of a_0. Model 1 captures the variation in the volatility arising from the asset price. Models 2 and 3 capture additional variation in volatility due to time variation. Dumas et al. (1998:2069) chose quadratic forms for the volatility function, mainly due to the Black-Scholes S&P 500 options' implied volatilities having a parabolic shape. They used S&P 500 index option data for the period June 1988 to December 1993. Their results showed the Root Mean Squared Valuation Error for

[5] Dumas et al. (1998) solved equation (2.4.1.2) with the Crank-Nicholson finite-difference method.

Model 0 was twice as much as Model 1. This means the model that included variation for volatility resulted in fewer errors than the constant volatility of the Black-Scholes model.[6]

Beber (2001) applied the models developed by Dumas et al. (1998) on the Italian stock market index between 1995 and 1998. He found the Mib30 stock index gave similar smirks as the S&P500 stock index. Two models suggested fitting the market data. Model 1, which is a linear model, and a quadratic model 2. These two models are defined as

Model 1:

$$\sigma = \beta_0 + \beta_1 K + \epsilon$$

Model 2:

$$\sigma = \beta_0 + \beta_1 K + \beta_2 K^2 + \epsilon$$

In the models stated above, Beber (2001:15) defines K as the moneyness and not absolute strike like Dumas et al. (1998). The last term, ε, is the error arising from each model. His interpretations of the three parameters are as follows

- β_0 represents a general level of volatility which localises the implied volatility function. It can be seen as the constant of linear regression.
- β_1 is the coefficient that controls the displacement of the origin of the parabola with respect to ATM options. This parameter captures the negative profile that is responsible for the asymmetry in the risk-neutral probability density function.
- The extra parameter, β_2, in model 2 provides the implied volatility function a degree of curvature.

2.4.2 Quadratic Function for the ALSI Volatility Surface

Kotzé et al. (2009:14) suggested fitting the same quadratic function as Beber (2001) to model implied volatility surfaces for ALSI implied volatility data. They fitted three parameters to the data following a principle component analysis done by Alexander (2008:257). She found that the dynamics of the volatility skew from FTSE 100 index options are driven by three factors, namely, parallel shifts, tilts and curvature. The model Kotzé et al. (2009:14) proposed is as follows

.

$$\sigma_{model}(\beta_0, \beta_1, \beta_2) = \beta_0 + \beta_1 K + \beta_2 K^2 \tag{2.4.2.1}$$

The variables in equation (2.4.2.1) are defined as

[6]See Table VII in Dumas et al. (1998:2102) for all estimation results.

- K is the strike price divided by the spot price, also more commonly known as the moneyness.
- β_0 is the constant volatility parameter, responsible for the shift or trend with $\beta_0 > 0$. Take note of the fact that $\sigma \underset{K\to 0}{\longrightarrow} \beta_0$.
- β_1 is the correlation parameter, responsible for the slope of the model. In order for the no-spread-arbitrage condition to be satisfied, $-1 < \beta_1 < 0$.
- β_2 is defined as the volatility of volatility ("vol of vol") parameter which accounts for the curvature or convexity in the model. The no-calendar-spread arbitrage condition requires $\beta_2 > 0$.[7]

2.4.3 Volatility Term Structure

The volatility term structure is the variation in implied volatility due to a variation in time to maturity of the options. Volatility surfaces combine volatility smiles with the term structure to tabulate the volatilities appropriate for pricing an option with any strike price and any maturity (Hull, 2012:416). Equation (2.4.2.1) is independent of time; it must therefore be combined with the ATM volatility term structure to generate a volatility surface. Kotzé (2009:15) optimised the deterministic volatility function and ATM volatility term structure separately, for each expiry date. The functional form for the ATM volatility term structure is

.

$$\sigma_{atm}(\tau) = \frac{\theta}{\tau^{\lambda}} \tag{2.4.3.1}$$

Here,

- τ is the months to expiry, calculated by $\frac{date_{expiry} - date_{start}}{365} \times 12$
- λ determines the slope. If $\lambda > 0$, the ATM volatility term structure will be downward sloping whereas $\lambda < 0$ implies a downward sloping ATM volatility term structure.
- The parameter θ controls the short-term ATM volatility curvature.

2.4.4 Quadratic Approach with no-arbitrage across time

To ensure no-arbitrage opportunities arise across time, which is across different expiry dates, the following smoothing function was developed by Kotzé et al. (2013:386)

$$\min_{\theta_k, \lambda_k} \left\| \beta_k(\tau) - \frac{\theta_k}{\tau^{\lambda_k}} \right\|^2 \tag{2.4.4.1}$$

---

[7]See Fengler (2005) and Gatheral and Jacquier (2013) for information on no-spread- and no-calendar-spread arbitrage.

Here $k = 0, 1, 2$ and represents each of the three parameters from formula (2.4.2.1). Therefore Formula (2.4.3.1) must be minimised in order to obtain a curvature parameter, θ_k, as well as a slope parameter, λ_k, for each of the parameters in the parameter triplet $(\beta_0, \beta_1, \beta_2)$. The expiry dates are defined as months to expiry and denoted as $\tau = \frac{date_{expiry} - date_{start}}{365} \times 12$. From here the parameter term structure is defined by Kotzé et al. (2013:386) as

$$\beta_k^{\tau}(\theta_k, \lambda_k) = \frac{\theta_k}{\tau^{\lambda_k}} \tag{2.4.2.2}$$

Substituting each of the parameters from formula (2.4.2.2) into formula (2.4.2.1) results in the fully described volatility surface. Therefore, the fully described Quadratic Deterministic Volatility surface is defined by six parameters as

$$\sigma(K, \tau)^{model} = \frac{\theta_2}{\tau^{\lambda_2}} K^2 + \frac{\theta_1}{\tau^{\lambda_1}} K + \frac{\theta_0}{\tau^{\lambda_0}} \tag{2.4.2.3}$$

with

- $\frac{\theta_0}{\tau^{\lambda_0}} > 0,$
- K the moneyness, i.e. $K = \frac{Strike}{Spot}$.

Kotzé et al. (2009:32) further noted that each of the parameter pairs (θ_k, λ_k), $k = 0, 1, 2$, implies that the parameters mean-revert, which means they tend to their average in the long term.

2.5 SUMMARY

The formulas to price options using the Black-Scholes-Merton pricing formulas were defined in this chapter. Assumptions regarding constant volatility led to the development of alternative models which captures the fact that volatility in itself varies. The SABR model, described in Chapter 2.3, introduced a stochastic process which includes an extra term of randomness. The extra term is a stochastic process that captures the so-called volatility of volatility. Furthermore, a formula to calculate the volatility implied by the SABR model was defined as well as the literature needed to estimate the parameters in the model.

Another approach to estimate volatilities in the South African market is using deterministic models. These models require no assumptions of the dynamics of the underlying process. In this chapter the literature leading up to the Quadratic Deterministic model was discussed.

The following chapter contains the methodology used to implement the SABR model and Quadratic Deterministic model in the South African market. The literature from this chapter is used to explain the different methods in estimating the implied volatility surfaces of the South African ALSI.

CHAPTER 3
RESEARCH METHODOLOGY

3.1 INTRODUCTION

This chapter presents the methodology used to implement the SABR model and the Quadratic Deterministic approach for the South African Market. Detailed explanation is given on the models, with less detail on the data used and the process of cleaning this data, as the main focus of this study is the implementation of these two models in the South African market. The first part of this chapter sets out the methodology used for the SABR model, the second part for the Quadratic Deterministic model and the third part for the comparison methods of the two models in question. A description of the data used is also given in short.

3.2 DESCRIPTION OF DATA

In South Africa, index options traded on the ALSI are listed on the Equity Derivatives Market, formerly SAFEX. Kotzé et al. (2009:17) pointed out that although most index options are traded on the ALSI, it can still be illiquid at times with few ATM trades. This shortage in trade data may lead to significant errors in calibration of volatility models.

Raw historical data for 2014 was extracted from the JSE's website.[8] These Excel files contain the equity contracts traded on each specific day. The data is first cleaned and grouped together into one spreadsheet containing all of the raw data needs. For the purposes of this text, only ALSI contracts are used with expiries in March 2014, June 2014, September 2014 and December 2014. Trade data in the final spreadsheet includes the strike price, spot price and traded volatility. The more liquid contracts are that of March, thus the contract with the earliest expiry dates. As time to expiry gets longer, the illiquidity of that contract increases.

3.3 THE SABR MODEL

The methodology to implement the SABR model defined in Chapter 2.3 now follows. Estimation of the parameters are all done in Matlab, by using and adapting the built-in Financial Instruments Toolbox functions.

3.3.1 Estimation of β

First the parameter β can be estimated using a log-log plot of $\sigma(F,F)$ and F from the following function of Hagen et al. (2002:91)

[8]The source, https://www.jse.co.za/downloadable-files?RequestNode=/Safex/EdmStats, can be used to obtain all data sets.

$$\ln \sigma_B(F,F) = \ln \alpha - (1-\beta)\ln F + \cdots \tag{3.3.1}$$

From this, linear regression of the observed ATM volatilities and forward rates can be used to estimate the slope. Tran et al. (2014:40) suggested β being the slope of the line above the estimated slope. That is the estimated slope plus one.

This study however followed the second approach suggested by West (2005:12), in which economic considerations in the South African context led to a specific, pre-selected value for β. The parameter β is pre-specified as $\beta = 0.7$ from which the rest of the parameters are estimated, respectively. He then chose the model that produced the smallest squared errors. Hagen et al. (2002:91) showed pre-selecting β does not result in a substantial difference in the quality of the fit. With the value of β pre-selected, the rest of the parameters can now be estimated. The following subsection sets out the methods in doing this.

3.3.2 Estimation of α_0, ρ and ν

First, consider only the parameter α_0. There are two ways to estimate α_0. First estimation is done by obtaining an implied α_0 from σ_{ATM} and secondly by minimising the objective equation (2.3.2.1).

In order to estimate α_0 from σ_{ATM}, first assume the values of ρ and ν are known inputs. As mentioned above, use the following equation to obtain a cubic function of α_0

$$\sigma_{ATM} = \sigma_B(f,f) = \frac{\alpha_0}{f^{(1-\beta)}}\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha_0^2}{f^{2-2\beta}} + \frac{1}{4}\frac{\rho\beta\alpha_0\nu}{f^{(1-\beta)}} + \frac{2-3\rho^2}{24}\nu^2\right]T\right\} \tag{3.3.2}$$

Manipulating equation (3.3.2) by first multiplying with $f^{(1-\beta)}$ through the whole equation, then subtracting both sides σ_{ATM} results in the following expression

$$0 = A\alpha_0^3 + B\alpha_0^2 + C\alpha_0 - \sigma_{ATM} f^{(1-\beta)}$$

where

$$A = \left[\frac{(1-\beta)^2 T}{24 f^{(2-2\beta)}}\right], B = \left[\frac{\rho\beta\nu T}{4 f^{(1-\beta)}}\right] \; and \; C = \left[1 + \frac{2-3\rho^2}{24}\nu^2 T\right]$$

West (2005:7) pointed out that one would choose the smallest non-negative real root if there are more than one real root. The Matlab function *roots()* is used to obtain the roots of the cubic function.

The second approach is described by Hansen (2011:51) as the free approach. The free approach estimates α_0, ρ and ν by minimising the objective equation (2.3.2.1).[9]

By using the first approach, only ρ and ν need to be estimated. These parameters are regarded as inputs from which a model error expression is determined. The error expression measures the distance between the traded volatilities and the volatilities implied by these parameters. As West (2005:8) did, minimise the error expression among all of the input pairs of ρ and ν.

3.3.3 Implementation of the SABR model

Implementation is done in Matlab. Nkounga (2015:122) used the functions and code for the SABR model written by Fabrice Douglas Rouah.[10] Appendix A contains the coding used to implement the SABR model in Matlab.

3.4 QUADRATIC DETERMINISTIC APPROACH

Recall from chapter 2.4 the model defined as

$$\sigma_{model}(\beta_0, \beta_1, \beta_2) = \beta_0 + \beta_1 K + \beta_2 K^2. \qquad (2.4.2.1)$$

The method we used to estimate volatility skews with the quadratic deterministic approach is the methodology developed by Kotzé et al. (2009:27). We start with initial values for the parametersβ_0, $\beta_1$andβ_2, called the parameter triplet. The parameter triplet is then found such that the Euclidean distance between the model volatilities and the traded volatilities are minimised. At time t_0 the minimisation problem to be solved is

$$\min_{\beta_0,\beta_1,\beta_2} \left\| \sigma_{t_i}^{model} - \sigma_{t_i}^{traded} \right\|^2 \; with \; t_i \; \in [t_0 - h, t_0] \qquad (3.4.1)$$

The minimisation problem is solved subject to the constraints

- $\beta_1, \; \beta_2 > \; 0$
- $-1 \; < \beta_0 < \; 0$

The parameters are calculated for each ALSI expiry. This is done by using the *lm()* function in R, which is a regression method. Thus, for each expiry we get a parameter for the slope (β_0), shift (β_1) and volatility-of-volatility (β_2). The ATM volatility generated by the model can now be calculated as

$$\sigma_{ATM}^{model}(\beta_0, \beta_1, \beta_2, \tau) = \beta_0(\tau) + \beta_1(\tau) + \beta_2(\tau) \qquad (3.4.2)$$

By using equation (2.4.2.1) and equation (3.4.1) we obtain the floating volatility skews as follows

---

[9]West (2005:8) and Hansen (2011:51) tested both approaches and preferred the paramatisation approach, which this paper therefore also followed.

[10]Source of codes can be found at http://www.volopta.com

$$\sigma_{float}^{model}(\tau) = \sigma_{model}(\tau) - \sigma_{ATM}^{model}(\tau)$$
$$= \beta_1^{\tau}(K-1) + \beta_2^{\tau}(K^2-1) \quad (3.4.3)$$

Kotzé et al. (2009:20) pointed out that although equation (3.4.2) results in the ATM model volatilities and equation (3.4.3) the floating volatilities, the correct absolute volatilities are needed. This is obtained by minimising the function

$$\min_{\theta,\lambda} \left\| \sigma_{ATM}^{model}(\tau) - \frac{\theta}{\tau^{\lambda}} \right\|^2 \quad (3.4.4)$$

where τ is the months to expiration.

From Chapter 2.4.4, expression (2.4.4.1) below is the minimisation problem solved for each of the three beta parameters. The *optim()* function in R is used to obtain these six parameters needed to fully describe the Quadratic Volatility surface.

$$\min_{\theta_k,\lambda_k} \left\| \beta_k(\tau) - \frac{\theta_k}{\tau^{\lambda_k}} \right\|^2 \quad (2.4.4.1)$$

These six parameters (θ_k, λ_k), $k = 0, 1, 2$, are directly substituted into formula (2.4.2.3) below, which fully describes the Quadratic Deterministic volatility surface.

$$\sigma(K,\tau)^{model} = \frac{\theta_2}{\tau^{\lambda_2}}K^2 + \frac{\theta_1}{\tau^{\lambda_1}}K + \frac{\theta_0}{\tau^{\lambda_0}} \quad (2.4.2.3)$$

Kotzé et al. (2015:64) extended formula (2.4.2.3) such that it consists of three inputs, namely the strike price, the spot price and the time to maturity. The following formula can be used to price the volatility for any of the specified inputs, given the process followed in this chapter to obtain the six parameters describing the model

$$\sigma(F,K,\tau)^{model} = \frac{\theta_2}{\tau^{\lambda_2}}\left(\left(\frac{K}{S}\right)^2 - 1\right) + \frac{\theta_1}{\tau^{\lambda_1}}\left(\frac{K}{S} - 1\right) + \sigma_{ATM}(\tau) \quad (3.4.5)$$

where

- K the strike price[11],

[11]Note K is not the moneyness anymore as previously defined.

- S the spot price,

and,

$$\sigma_{ATM}(\tau) = \frac{\theta_2}{\tau^{\lambda_2}} + \frac{\theta_1}{\tau^{\lambda_1}} + \frac{\theta_0}{\tau^{\lambda_0}} \tag{3.6.6}$$

Formula (3.4.6) results from setting $K = 1$ in Formula (2.4.2.3). Thus when the option is at-the-money, the strike price is equal to the spot price; hence the moneyness is equal to one.

3.5 COMPARISON METHODOLOGY

Comparison of the two models is done graphically and statistically. The volatilities implied by the two models are plotted against the market volatilities to compare visually if the models fit the market data. A goodness-of-fit is done by calculating the MSE of each model and comparing the results to obtain a definitive conclusion on which model is more suitable for the illiquid South African market. Furthermore, a R^2 value is calculated for each model to check how close the market data is when compared to the fitted regression line. The direct correlation measure is also calculated to measure the degree and direction which the models and the market volatilities move.

The comparisons are done for the ATM volatilities, volatility skews and term structure for two separate dates. Using more than one date assures the validity of the two models.

3.6 SUMMARY

In this chapter the methodology used to implement the two models in question, which was described in Chapter 2, were discussed. The method to fit and obtain the necessary parameters was first discussed for the SABR model, then for the Quadratic Deterministic approach. A description of the comparison methods was also given. The following chapter sets out the results obtained from the implementation of the methodology from this chapter.

CHAPTER 4
FINDINGS

4.1 INTRODUCTION

This chapter will present and discuss the results obtained from fitting the two models to the market data using the methodology described in Chapter 3. Modelling is done by fitting the data from two separate dates. The first date is 19 December 2013 and the second date is 19 March 2014. Throughout this chapter, analysis is first done and explained for 19 December 2013 and then for 19 March 2014.

4.2 MARKET VOLATILITY SKEWS

The ALSI volatility skew for four maturity dates are given in Figure 4.1, Figure 4.2, Figure 4.3 and Figure 4.4. As suggested by Hull (2012:414), the volatility skews for equities should be downward sloping. The implied volatility used in the market is the volatility implied by Black-Scholes model.

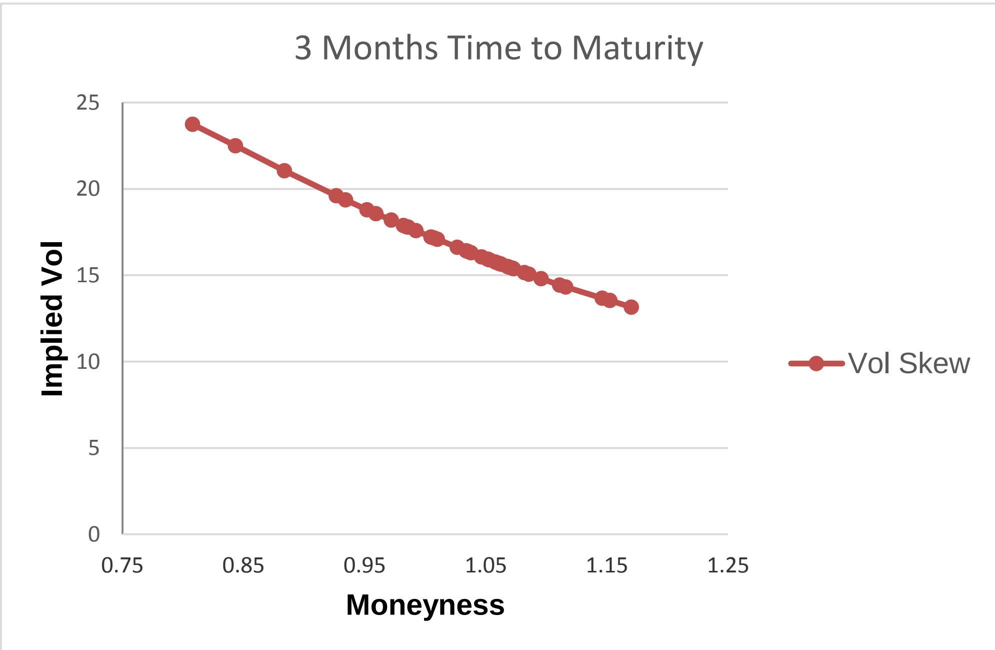

Figure 4.1: Volatility skew of an ALSI call option, 3 months to maturity.

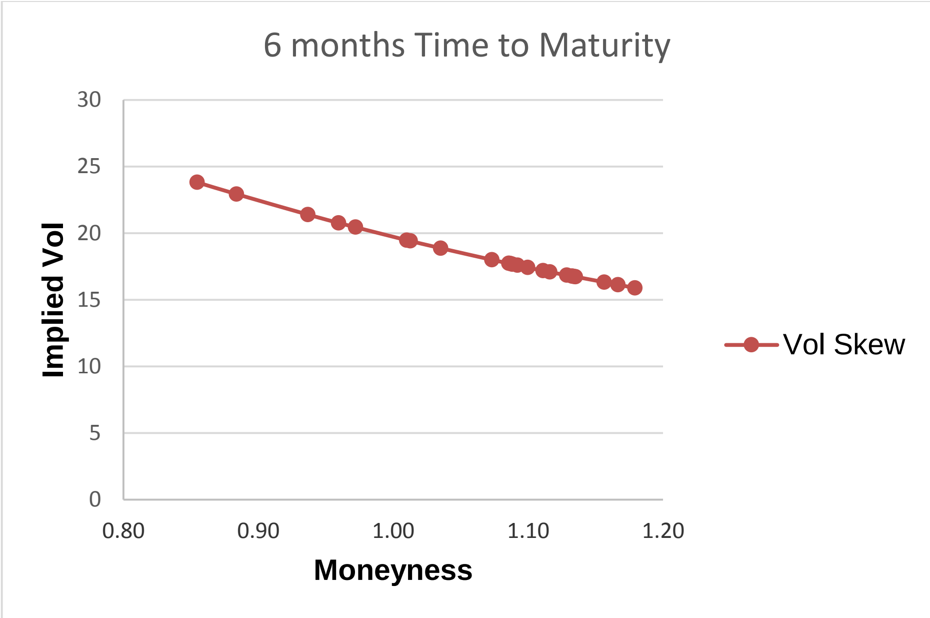

Figure 4.2: Volatility skew of an ALSI call option, 6 months to maturity.

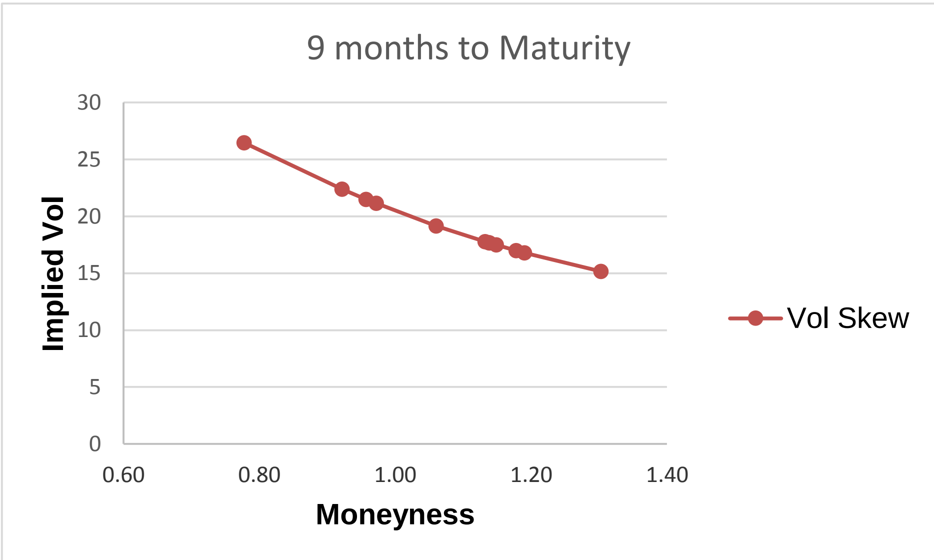

Figure 4.3: Volatility skew of an ALSI call option, 9 months to maturity.

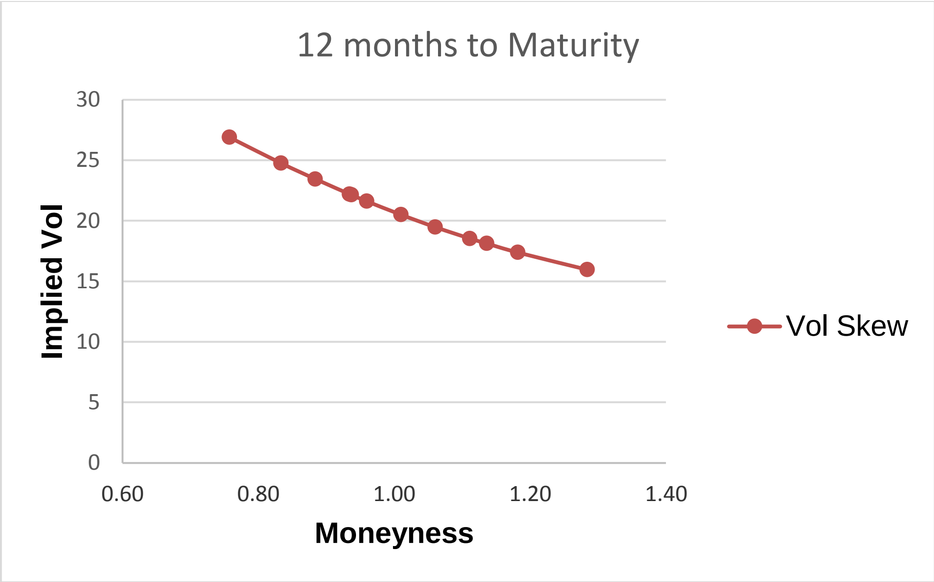

Figure 4.4: Volatility skew of an ALSI call option, 12 months to maturity.

The plots show that low strike options have higher implied volatilities than high strike options. The reason for this downward sloping volatility skews, which backs up the results obtained, is described by Figlewski and Wang (2000) as the leverage effect. Figlewski et al. (2000:22) refers to the leverage effect as the link between stock returns and volatility changes. They found that only changes in the stock's price have significant changes in the volatility of that stock.

Hull (2012:415) explains the reason for a downward sloping smile in equity options concerns leverage in a way that as a company's equity declines in value, the company's leverage increases. This in turn means the company becomes riskier, which leads to an increase in volatility. The opposite effects are true for an increase in a company's value.

An example of the cleaned data input used for the results in the following sub chapters is found in Table 4.2.1 below. Table 4.2.1 represents only the data used for a call option expiring on 18 December 2014. The data has to be sorted for each expiry data. Appendix C represents a screenshot of the raw data.

Moneyness	Market Volatility
0.7574	0.2689
0.8332	0.2476
0.8837	0.2345
0.9342	0.2220
0.9367	0.2214
0.9594	0.2161
1.0099	0.2050
1.0604	0.1947
1.1109	0.1854
1.1362	0.1812
1.1816	0.1738
1.2839	0.1596

Table 4.2.1: Data input on 19 March 2014 for a call option expiring on 18 December 2014

4.2 THE SABR MODEL

Implementation of the SABR model using the methodology described in Chapter 3.3 is done by utilising the built in Matlab functions in Matlab's Financial Instruments Toolbox. Appendix A contains a full copy of the codes and functions used, which was adapted from the Matlab functions to suit the South African market and data. Parameters for each of the four expiry dates are given in table 4.1 below. The value of β is chosen initially as $\beta = 0.7$. West (2005:12) found that this value for beta suits the South African market, after taking a certain amount of economic factors into consideration. These parameters can be used in formula (2.3.1) to price the volatility of any strike given the specific maturity date's parameters.

Months To Expiry	Alpha (α)	Beta (β)	Rho (ρ)	Nu (ν)
3	1.0339	0.7	-0.5930	0.86244
6	1.1774	0.7	-0.6823	0.64985
9	1.1737	0.7	-0.6327	0.60349
12	1.2454	0.7	-0.6954	0.53416

Table 4.1: Estimated SABR parameters for 19 Dec 2013

The parameters in Table 4.1 can be used to calculate the ATM volatility term structure by substituting them into formula (2.3.2).[12] Four ATM volatilities from the SABR model can be compared to the only four ATM volatilities obtained from the market. This gives an indication of how well the model fits the market data. The SABR model results in a MSE of 0.00294% and a R^2 of 0.95957 when compared to the market. An R^2 value of 0.95957 is close to one, which would suggest that the model reflects almost all of the variance from the implied volatilities from the market. The direct correlation between the ATM SABR volatilities and the ATM market volatilities is 0.97957. These results together with Figure 4.5 below indicate that the SABR model is a good fit for the market data.

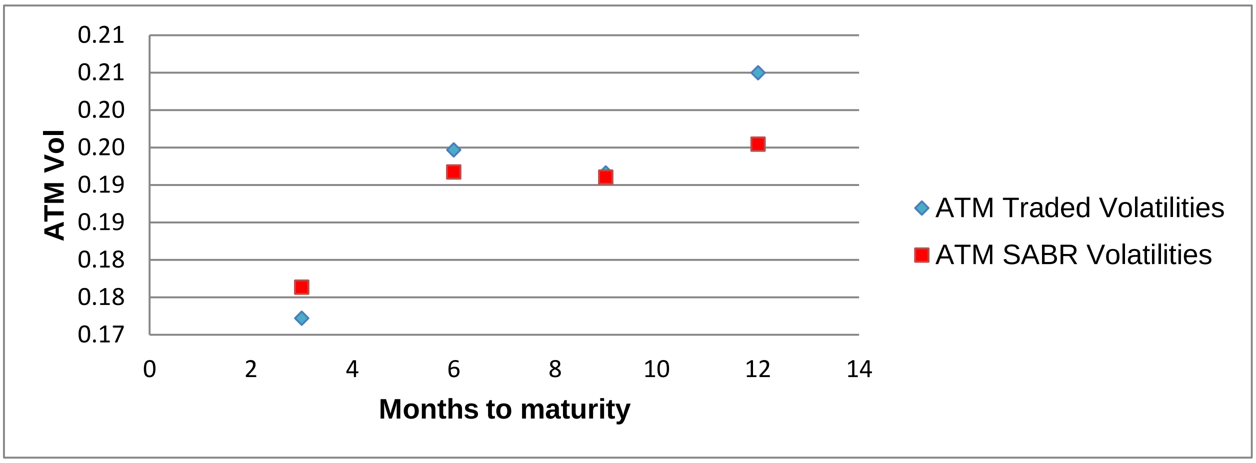

Figure 4.5: Comparison of ATM traded volatilities and ATM SABR volatilities on 19 December 2013

Using the parameters for 3 months to expiry ($\alpha = 1.0339$, $\beta = 0.7$, $\rho = -0.593$ and $\nu = 0.86244$) and Formula (2.3.1), the SABR volatility can be obtained and compared to the market volatility. After shifting the SABR volatilities with the average difference between the SABR volatilities and Market volatilities, the adjusted SABR volatilities are obtained. This results in an MSE of 0.00072% when the volatilities from the SABR model are compared to the market volatilities. Figure 4.6 below shows the plot of this comparison. The volatilities implied by the SABR model over-estimates the volatility, although the shape of the curve seems to be in line with the market volatilities. This research assignment suggests taking the mean difference between each data point and subtracting it from the SABR volatilities. Figure 4.6 shows the adjusted SABR model to then be a reasonable fit of the market data.

[12]Hagen et al. (2002:90) found that higher order terms in formula (2.3.2) are unnecessary as it does not lead to significantly more accurate results.

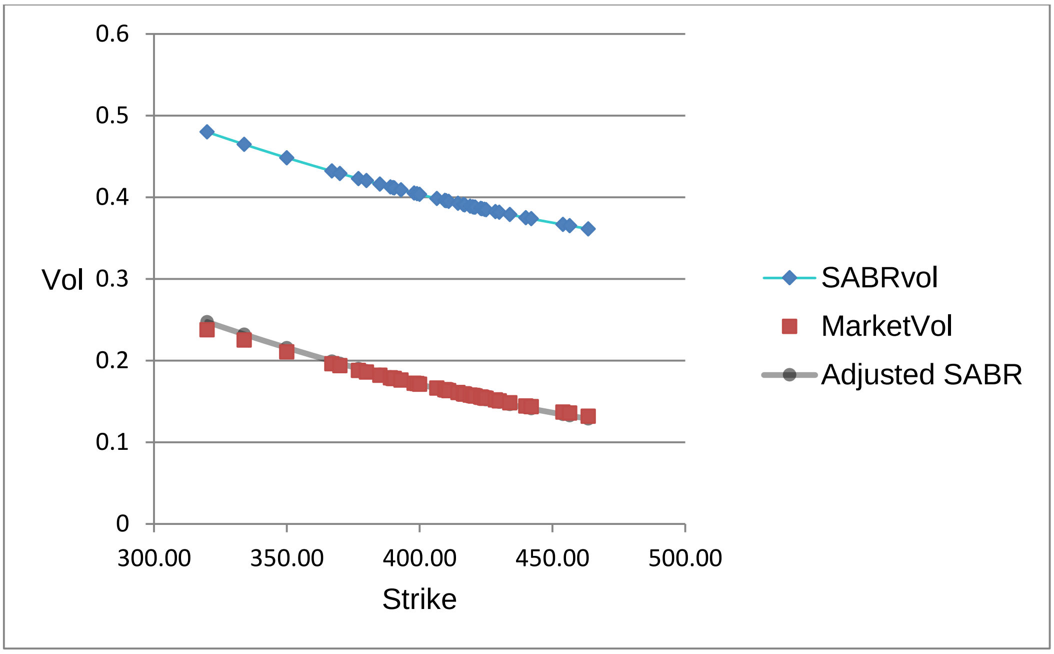

Figure 4.6: SABR model volatilities vs. Market volatilities on 19 December 2013

The effectiveness of the adjustment proposed by this paper can be seen in Figure 4.6 above. The average error on each pair of volatilities, SABR- and Market volatilities, for each strike is calculated and subtracted from the original SABR volatilities. This seems to be an effective method to ensure the volatilities implied by the SABR model fits the market data. The reason behind the SABR model overestimating the market volatilities might be from formula (2.3.1). The extra terms, which falls away when the option is ATM, seems to add to the overestimation of the market volatility. However, the method suggested above addresses this problem sufficiently.

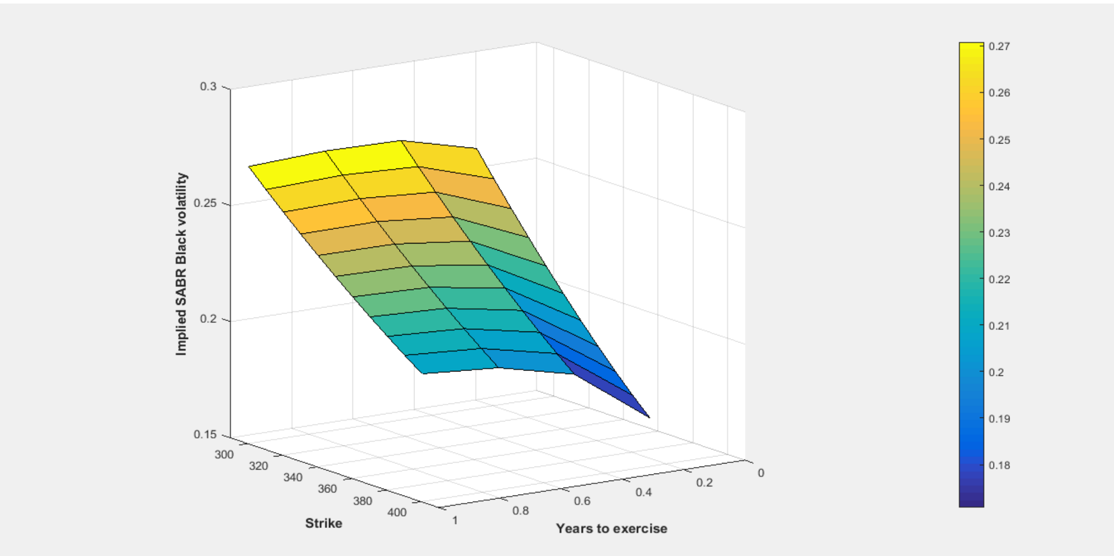

Figure 4.7: ALSI volatility surface given SABR model on 19 December 2013

The ALSI volatility surface can be obtained for any vector of strikes by using the parameters in Table 4.1. As an example, use strikes ranging from R300 to R400. As mentioned in Chapter 3, the Matlab functions *blackvolbysabr()* and *surf()* are used to graph the surface. Above, in Figure 4.7, a downward sloping concave volatility surface is obtained. Haugh (2009:4) describes this kind of volatility surface as a surface depicting times of market stress. That is that short term options have smaller volatilities than long term options. Figure 4.7 shows as time to maturity increases, the volatility increases.

As an example to show how any point on Figure 4.7 can be used to price the volatility, consider an option with a strike price of R350, a spot price of R396 and 3 months to maturity. The corresponding volatility results in 21.56%. Formula (2.3.1) can be used with the specified inputs and parameters for 3 months to expiry from Table 4.1 to obtain this value.

The same analysis is done using the current date as 19 March 2014 to check the reasonableness of the model for other dates. Table 4.2 below sets out the estimated SABR parameters, where the market data for four expiry dates are used. The four expiry dates are 20 March 2014, 19 June 2014, 18 September 2014 and 18 December 2014.

Days To Expiry	Alpha(α)	Beta (β)	Rho (ρ)	Nu (ν)
1	1.1168	0.7	-0.4859	2.1561
92	1.2041	0.7	-0.7371	0.7754
183	1.2139	0.7	-0.7477	0.6097
274	1.1980	0.7	-0.7020	0.5485

Table 4.2: Estimated SABR parameters on 19 March 2014

Formula (2.3.2) can now be used together with the parameters in Table 4.2 to calculate the ATM volatilities implied by the SABR model. The ATM volatilities from the SABR model are compared with the ATM volatilities from the market. Figure 4.8 below shows this comparison. The comparison results in a MSE of 0.000041%, which is a low measure of deviation from the actual ATM volatilities. A R^2 value of 0.965 is obtained when the ATM market volatilities are compared with the SABR ATM volatilities. The R^2 value is close to one. This gives an indication that the variance caused by the SABR model is reflected in the model. The direct correlation between the ATM model volatilities and ATM market volatilities is 0.982, which indicates the SABR model volatilities moves almost perfectly (direct correlation close to one) in the same direction as the ATM market volatilities. Figure (4.8), together with the error analysis shows the SABR model seems to be a reasonable fit as of 19 March 2014.

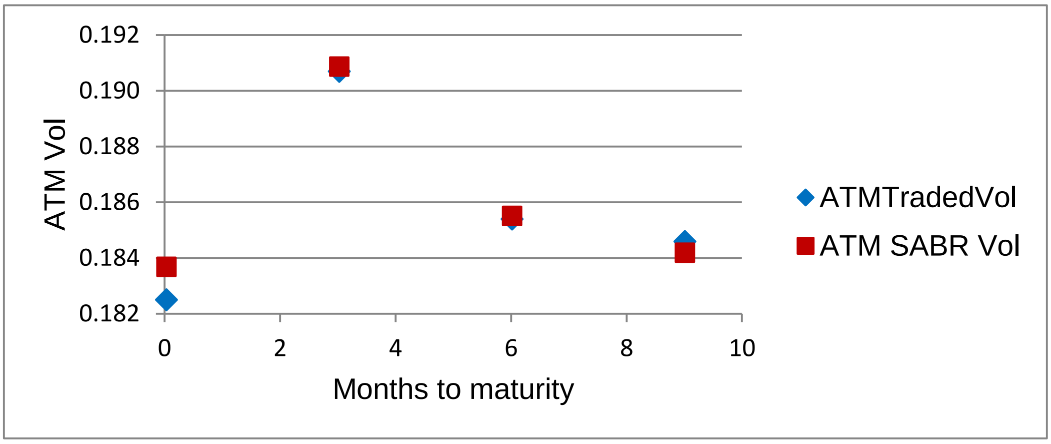

Figure 4.8: Comparison of ATM traded volatilities and ATM SABR volatilities for 19 Mar 2014

The volatilities from the SABR model can now be compared to the market volatilities for a specific term structure. As an example, consider a call option with one day left to expiry. This means the analysis is done on 19 March 2014 with expiry of the option the next day, i.e. 20 March 2014. The corresponding parameter values for an option expiring in one day can be found in the first row of Table 4.2. These parameter values are substituted into Formula (2.3.1) to obtain the SABR volatilities. The same strike prices are used as the corresponding market volatilities. This allows for direct comparison between the market volatilities and the SABR model volatilities. Figure 4.9 illustrates this comparison which results in a MSE of 0.00992%. The low MSE indicates the model fits the data reasonably well, although there is a higher MSE than that of the date 19 December 2013.

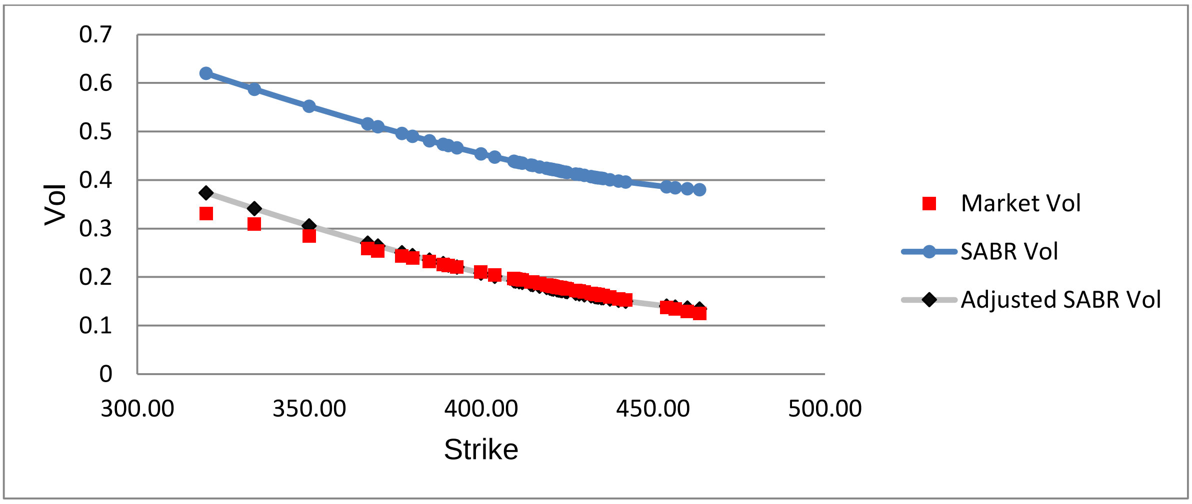

Figure 4.9: SABR model volatilities vs. Market volatilities on 19 March 2014

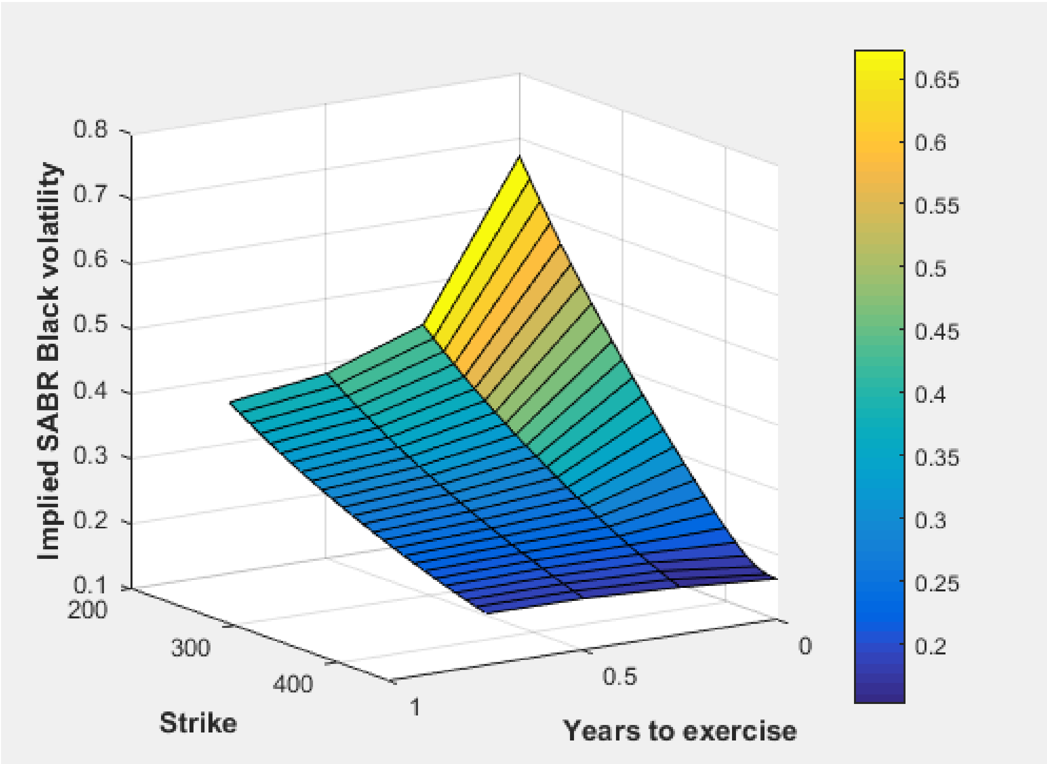

Figure 4.10:ALSIvolatility surface given SABR model on 19 March 2014

Figure 4.10 shows the volatility surface for a call option on the ALSI. It shows that options on the ALSI with lower strike prices tend to have higher volatilities implied by the SABR model. This is in conjunction with Haugh (2009:4) who noted that a principle feature of a volatility surface is that options with higher strikes tend to have lower volatilities implied by the model over the time to maturity. The volatility surface depicts the extremely high implied volatility for lower strikes and shorter time to maturity.

4.3 THE DETERMINISTIC QUADRATIC MODEL

Using the methodology described in Chapter 3.4, calibration is done in R for a call option with expiry date of 20 March 2014 and current date of 19 December 2013. The parameters were found to be: $\beta_0 = 0.7008$, $\beta_1 = -0.7663$, $\beta_2 = 0.2391$. Figure 4.11 shows the market data compared to the fitted function from equation (2.4.2.1). Parameters are optimised for each of the four expiry dates and listed in Table 4.3 below. Included in Table 4.3 are the ATM volatilities, which can be calculated using formula (3.4.2).[13]

[13] Note Tau is calculated as described in Chapter 2.4.3, thus it is calculated as the months to expiry.

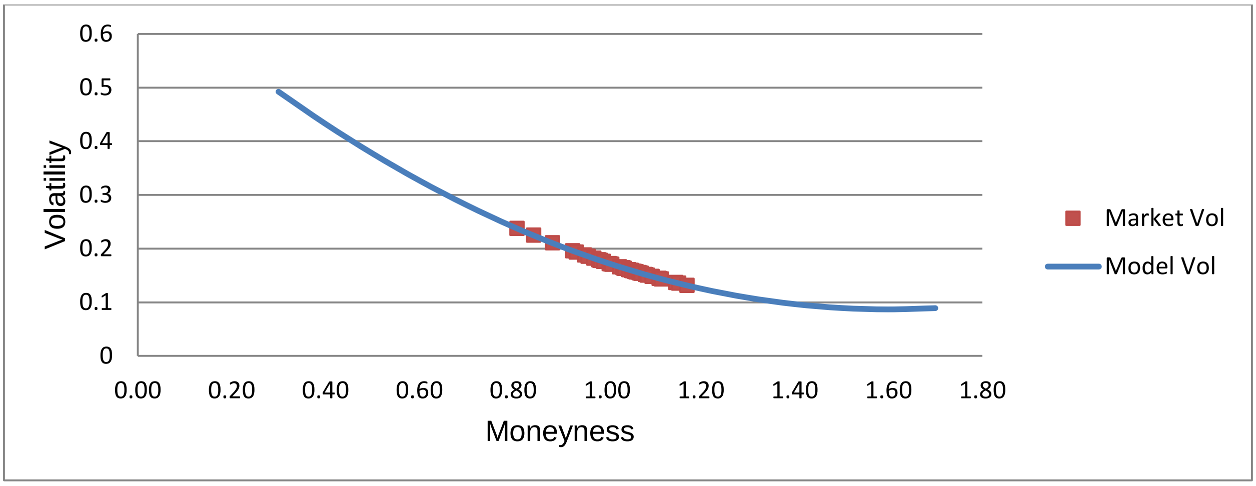

Figure 4.11: ALSI volatility skew on 19 December 2013 for option with 20 March 2014 expiry

Expiry	VolVol (β_2)	Slope (β_1)	Shift (β_0)	ATM Model Vol	Tau
20-Mar-14	0.2391	-0.7663	0.7008	0.1736	2.9918
19-Jun-14	0.2109	-0.6727	0.6590	0.1972	5.9836
18-Sep-14	0.1736	-0.5758	0.6071	0.2050	8.9753
18-Dec-14	0.1657	-0.5459	0.5874	0.2072	11.9671

Table 4.3: Regression Parameters and ATM model volatility on 19 December 2013

The floating volatilities can be calculated for each expiry by using formula (3.4.3). Figure 4.12 below shows the floating volatilities for each of the four expiry dates.

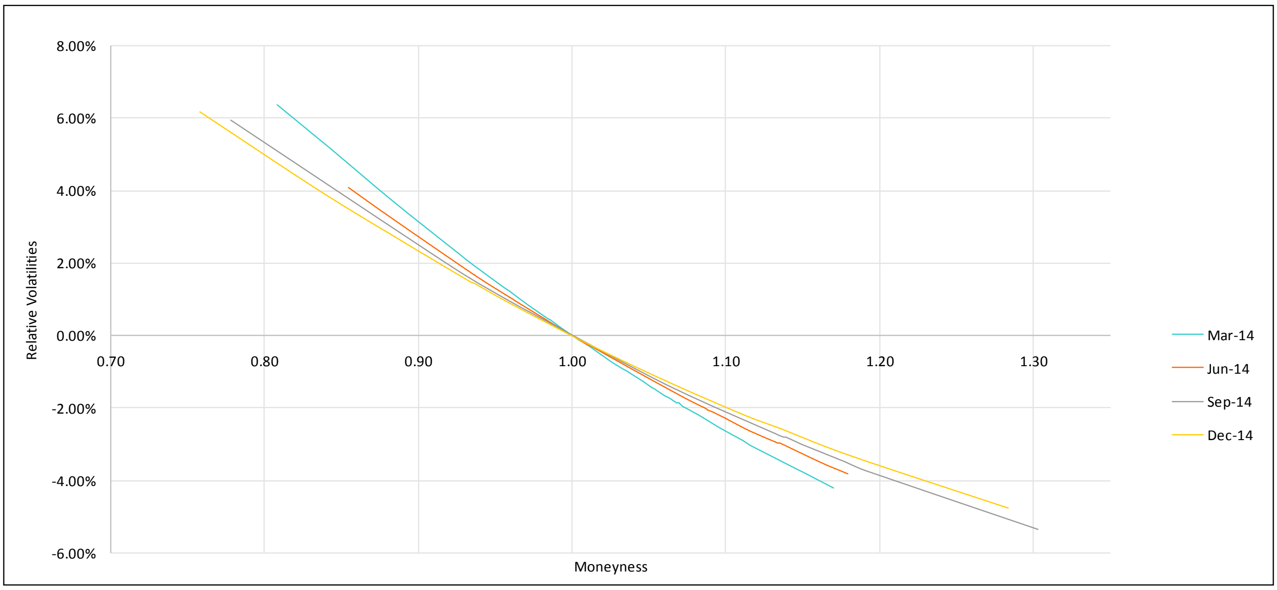

Figure 4.12: Deterministic floating skews for four expiry dates.

By optimising Formula (3.4.5) across all expiry dates, the final six parameters are obtained which fully describes the Quadratic Volatility Surface as set out in Chapter 3.4 and defined as in Formula (3.4.6). Table 4.4 below lists the six parameters, which is used to calculate the volatility for any specified moneyness and months to expiry. The calculated parameters are as of 19 December 2013 and can be recalibrated on a daily basis.

VolVol (β_2)		Slope (β_1)		Shift (β_0)	
θ_2	λ_2	θ_1	λ_1	θ_0	λ_0
0.32602	0.27088	-1.01552	0.248558	0.813264	0.129119

Table 4.4: Optimised parameters on 19 December 2013

Using formula (3.4.6) and the parameters in Table 4.4, the following equation can be used to determine the volatility for a call option on the ALSI given any moneyness and months to expiry:

$$\sigma(K,\tau)^{model} = \frac{0.32602}{\tau^{0.27088}}K^2 + \frac{-1.01552}{\tau^{0.24855}}K + \frac{0.81326}{\tau^{0.12912}} \quad (4.3.1)$$

As an example, if the volatility of a call option with three months left to expiry and a moneyness of 1.05 is required, Formula (4.3.1) results in a volatility of 16.11%. To compare the ATM traded volatilities with the ATM volatilities obtained from the model, substitute $K = 1$ in Formula (4.3.1) and use the corresponding months to expiry which is available in the market. Figure 4.13 below sets the comparison out graphically. The Quadratic Deterministic model gives a MSE of 0.004344%, which means the model is a reasonable fit to the market data.

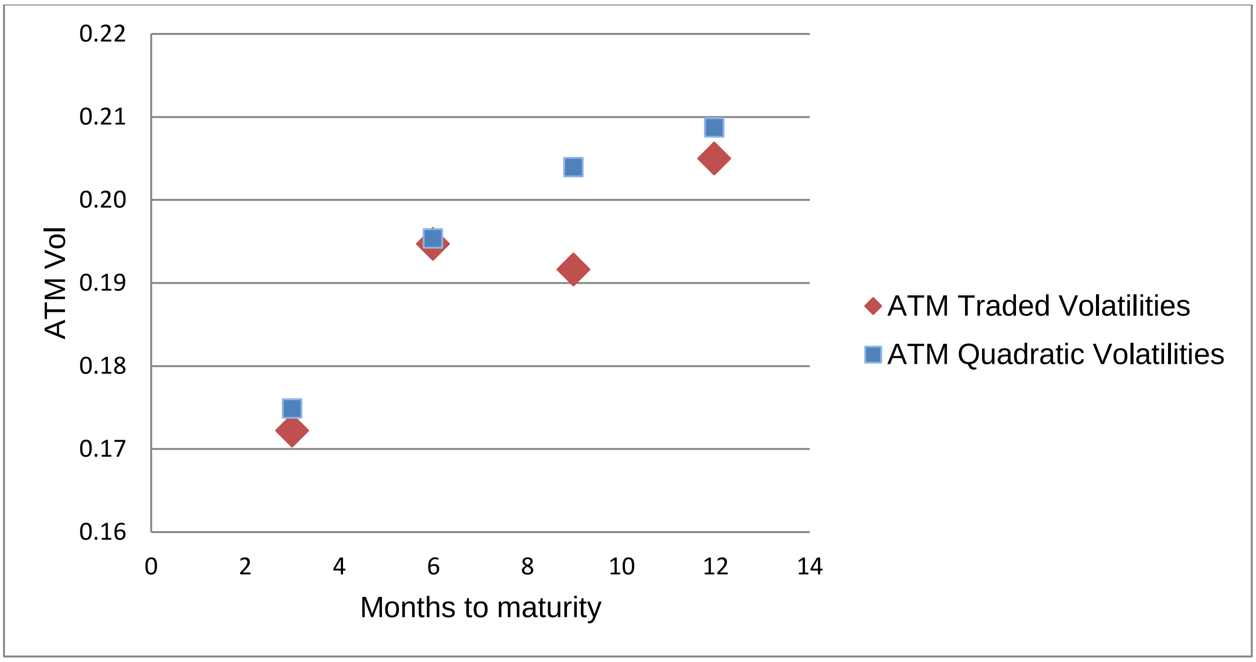

Figure 4.13: Comparison of ATM traded volatilities and ATM Quadratic model volatilities on 19 December 2013

In Figure 4.13, when comparing the third data point, the ATM volatility implied by the model is overestimating the ATM volatility implied by the market. This can lead to a potential problem in estimating volatilities. The Quadratic Deterministic model fits a quadratic function to the market data. The ATM Quadratic curve in Figure 4.13 shows the quadratic shape of the ATM volatilities implied by the model. Overall the model still seems to be a reasonable fit, but it is worthy to note that the model might have significant estimation errors on a particular rise or fall between successive times to maturity.

By using Formula (4.3.1) and, for example, 3 months to expiry, the volatility skew is calculated and compared to the market volatilities. The comparison results in a 0.0001336% MSE, which is again an indication that the model fits the data reasonably well. Figure 4.14 is a graphic representation of the comparison.

Formula (4.3.1) can now be used to calculate the volatility surface for an option on the ALSI index. As an example, the time to maturity ranges from 0.1 to 1 and the moneyness ranges from 0.8 to 1.5. This results in a volatility surface with time to maturity on the x-axis, moneyness on the y-axis and the volatility implied by the model on the y-axis. Figure 4.15 graphs the given example of the volatility surface.

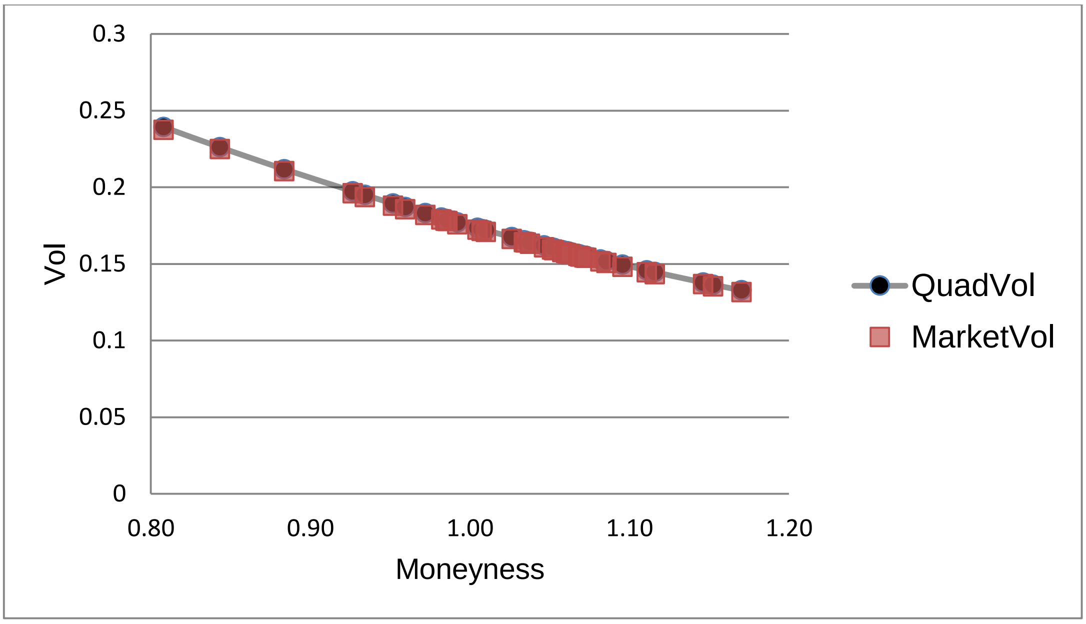

Figure 4.14: Quadratic Model Volatilities vs. Market Volatilities on 19 December 2013

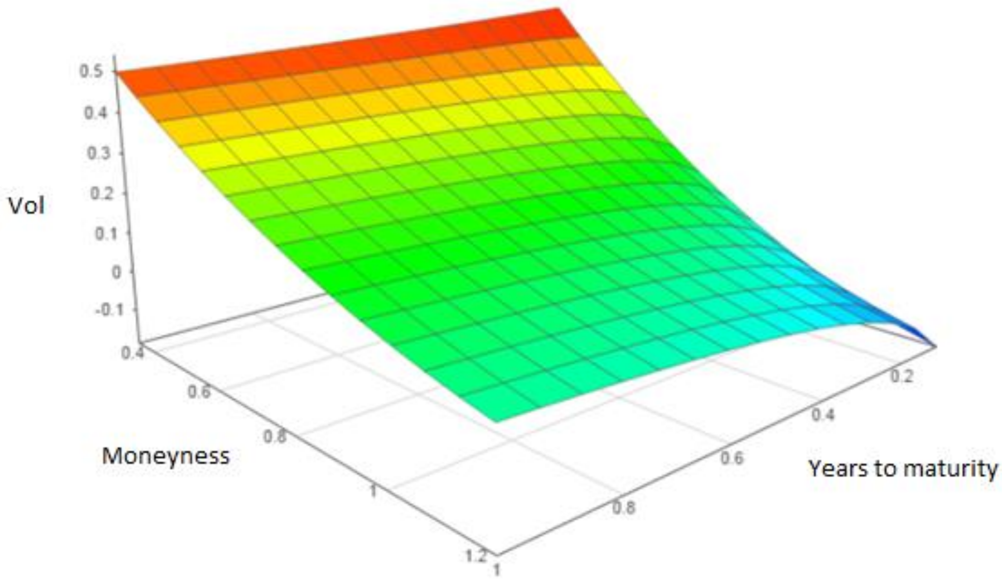

Figure 4.15: Quadratic Deterministic Volatility Surface on 19 December 2013

To test the reasonableness of the Quadratic model, another analysis is done for a different date, namely from 19 March 2014. As an example consider a call option with an expiry date of 19 June 2014. The beta parameters obtained are: $\beta_0 = 0.693208$, $\beta_1 = -0.67436$, $\beta_2 = 0.175585$. Substituting these beta parameters into formula (2.4.2.1) results in the model volatilities, given a moneyness ranging from 0.3 to 0.7. Figure (4.16) shows graphically how the initial model fits compared to the market volatilities.

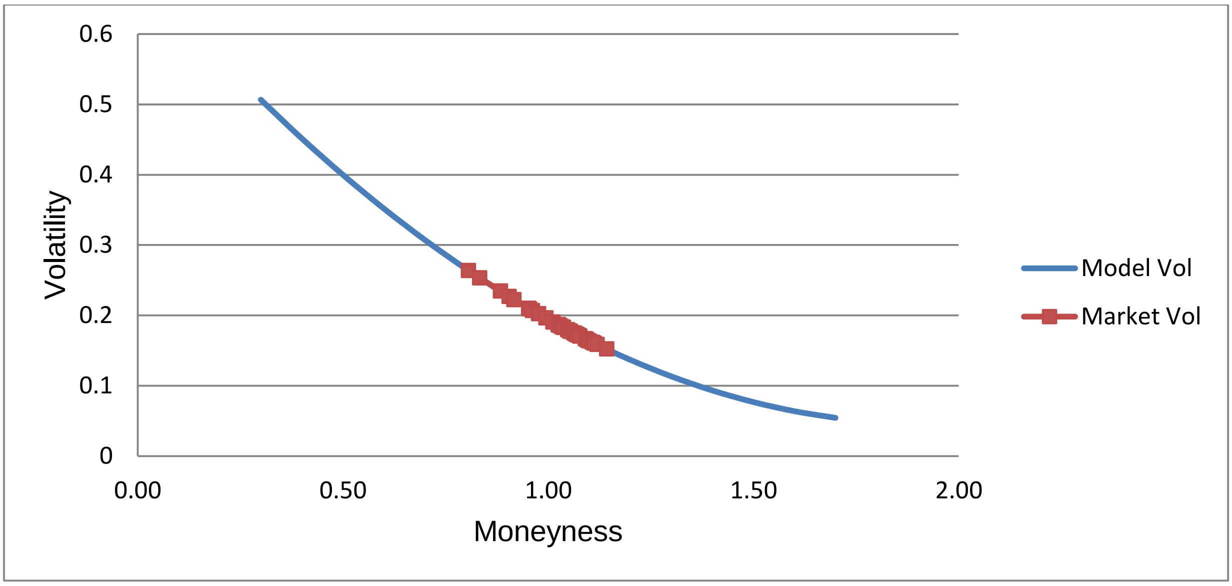

Figure 4.16: ALSI volatility skew on 19 March 2014 for call option with 19 June 2014 expiry

The beta parameters across all four expiry dates are listed in Table 4.5 below, as well as the corresponding ATM model volatilities. The same effect takes place as the time to maturity increases when comparing these parameters to the parameters in Table 4.3. That is the volvol and shift parameters decrease while the slope parameters increase as the time to maturity increases.

Expiry	VolVol (β_2)	Slope (β_1)	Shift (β_0)	ATM Model Vol	Tau
20-Mar-14	0.2064	-0.9884	0.9645	0.1825	0.0329
19-Jun-14	0.1756	-0.6744	0.6932	0.1944	3.0247
18-Sep-14	0.1623	-0.5804	0.6153	0.1972	6.0164
18-Dec-14	0.1505	-0.5208	0.5651	0.1948	9.0082

Table 4.5: Regression parameters and ATM model volatility as of 19 March 2014

Again, to find the final six parameters that completely describe the volatility surface, Formula (3.4.5) is optimised across all expiry dates. The optimised parameters are listed in Table 4.6 below.

VolVol (β_2)		Slope (β_1)		Shift (β_0)	
θ_2	λ_2	θ_1	λ_1	θ_0	λ_0
0.176186	0.048335	-0.701460	0.102119	0.721352	0.086547

Table 4.6: Optimised parameters on 19 March 2014

The parameters in Table 4.6, together with Formula (3.4.6), are used to obtain an equation that fully describes the volatility surface. Equation (4.3.2) below can therefore be used to estimate the volatility for any call option with the specified time to maturity and moneyness:

$$\sigma(K,\tau)^{model} = \frac{0.176186}{\tau^{0.048335}}K^2 + \frac{-0.701460}{\tau^{0.102119}}K + \frac{0.721352}{\tau^{0.086547}} \qquad (4.3.2)$$

For the sake of comparison, consider a call option with three months left to expiry and a moneyness of 1.05. Formula (4.3.2) results in a volatility of 18.18%. When $K = 1$ is substituted in Formula (4.3.2), the ATM model volatilities are obtained and can be compared to the ATM market volatilities. Figure (4.17) shows this comparison graphically. The Quadratic Deterministic model on the date of 19 March 2014 results in a MSE of 0.006063%. This is a low MSE which is an indication that the model is a good fit.

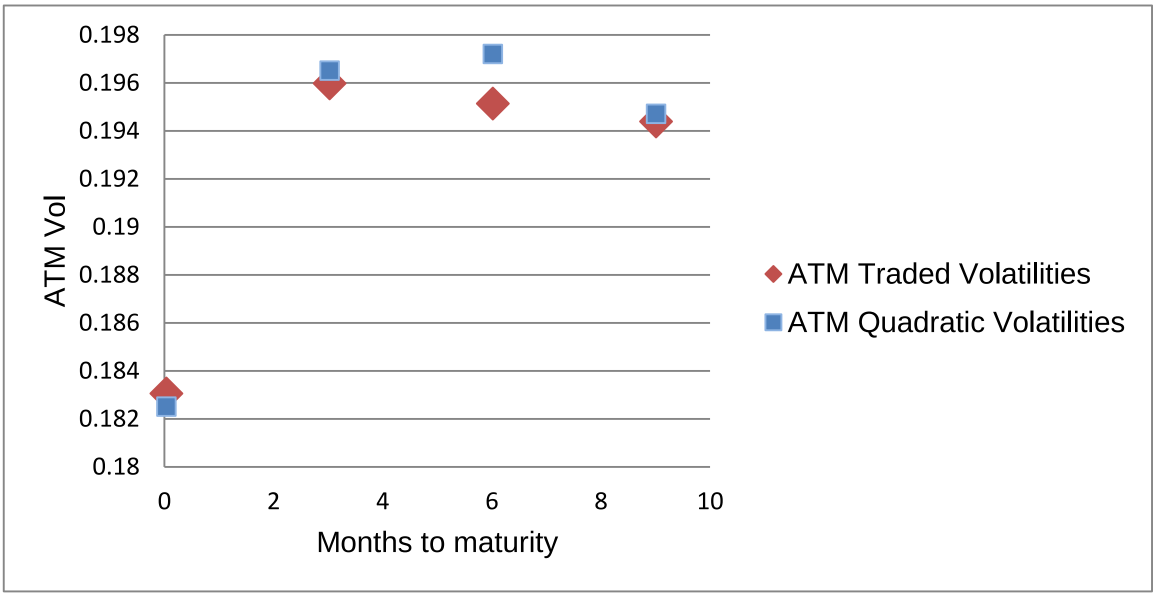

Figure 4.17: Comparison of ATM traded volatilities and ATM Quadratic model volatilities, as of 19 March 2014

The volatility skew is now calculated and compared with the market volatility to further assess how well the model fits. Once again formula (4.3.2) can be used to obtain the volatilities for a specified time to maturity. Figure (4.18) below shows the volatility skew for a call option with three months left to expiry. The market volatilities are shown on the same graph to once again compare how well the model fits the data. The comparison leads to a MSE of 0.0000396%.

Figure (4.18) is a further indication of the low MSE, as the market volatilities are superimposed onto the model volatilities. The volatility surface representation for a call option on 19 March 2014 can be found in Figure (4.19) below.

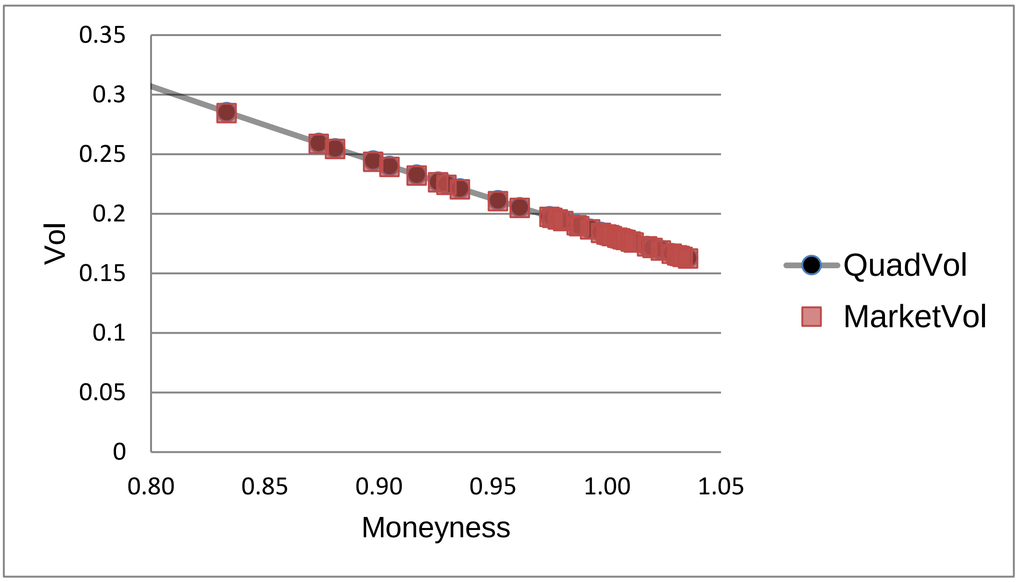

Figure 4.18: Quadratic Model Volatilities vs. Market Volatilities as of 19 March 2014

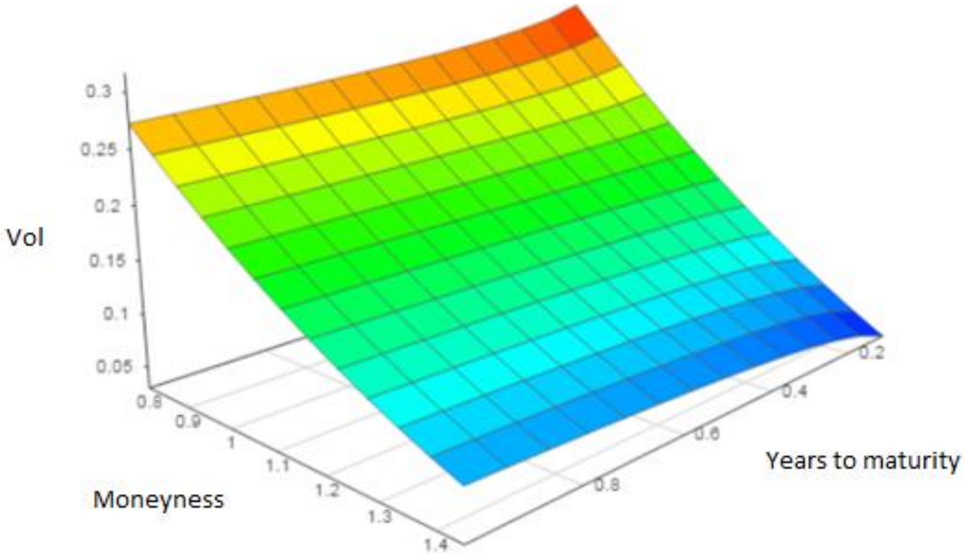

Figure 4.19: ALSI Quadratic Deterministic Volatility Surface on 19 March 2014

4.4 SUMMARY

In this chapter the results from implementing the two models in questions for ALSI market data are discussed. The volatility skews for each expiry was first shown to capture the relationship between the volatility implied by the market and the moneyness of the call option. A downward sloping volatility skew was observed for all expiries which depicts the leverage effect. The volatility skews also gave an indication that volatility is not constant. The constant volatility assumption in the Black-Scholes model leads to the necessity of alternative models which is able to capture the stochastic volatility from the market.

The SABR model was first implemented for two separate dates. This was done to check the reasonableness of the model on different dates. The original volatility estimates from the model overestimated the volatility when it is compared to the ALSI volatility. This text suggested adjusting the volatilities by the average mean of the errors, as the shape of the curve is the same as the market. The adjustment resulted in a MSE of 0.00072% on 19 December 2013 and a MSE of 0.00992% on 19 March 2014. This means that the SABR model, when adjusted is a reasonable method to estimate the volatility of volatility.

In the third part of the results, a Quadratic Deterministic approach was implemented for two separate dates. This is a simpler model of capturing the volatility of volatility by fitting a quadratic regression function to the ALSI data. The results from this model were satisfactory by fitting the data with a MSE of 0.00013% on 19 December 2013 and a MSE of 0.0000396% on 19 March 2014. Implementation of this model therefore estimates reasonable parameters, which can be used to estimate the volatility, and in turn the option prices, by fitting a quadratic function.

The conclusion of this research assignment follows in the next chapter. A recommendation is given on which model suits the illiquidity of the South African market the best to estimate the volatilities of the ALSI. Chapter 5 also includes a discussion on further research of this topic.

CHAPTER 5
CONCLUSION AND RECOMMENDATIONS

The purpose of this research assignment was to find methods which are able to capture the stochastic behaviour of volatility in an illiquid market like the South African ALSI market. In this research assignment two methods are discussed and compared, namely the SABR model by West (2005) and the Quadratic Deterministic model by Kotzé et al. (2009).

The previous literature that led up to each of the two models was discussed in Chapter 2. This enables the reader to understand where these models came from and how their underlying mathematics work, which is needed to implement the models. It was first necessary to review the Black-Scholes model as it assumes constant volatility. This allows the reader to understand why alternative models were later developed. For the SABR model it was necessary to explore the stochastic processes used to capture the volatility of volatility as well as the dynamics of the parameters. The literature regarding estimating the model parameters were also discussed. The literature review of the Quadratic Deterministic approach first includes a discussion on deterministic models. This is essential in understanding the reasoning behind a quadratic approach for the South African market. Chapter 2 concludes by setting out the optimisation functions to estimate the Quadratic Deterministic model's parameters.

The construction of the volatility models was presented in Chapter 3. This serves as the methodology chapter for the two models of this research assignment. A description of the ALSI market data was given followed by the different methods to implement the two models, respectively. The programming of the SABR model was done by adapting the Financial Instruments Toolbox available in Matlab and is presented in Appendix A. Programming for the Quadratic Deterministic approach was done in R by the author. The code to implement this model can be found in Appendix B.

Chapter 4 contains the results from implementing the SABR model and Quadratic Deterministic approach. The results showed that both models fit the ALSI market data reasonably well, although the SABR model first needed to be adjusted as it did not fit the data well initially. The SABR model over-estimated the volatilities. This research assignment adjusted the SABR implied volatilities by the average of the errors between each pair of SABR volatilities and market volatilities. The results showed that this approach leads to a reasonably good fit. The complexity of the SABR model when compared to the simplicity of the Quadratic Deterministic approach leads to a recommendation of implementing the Quadratic Deterministic model to estimate volatility surfaces in the South African market. Furthermore, the Quadratic Deterministic approach leads to a better fit of the market data on the separate dates used in this research assignment. The results from the SABR model showed that it does not fit the data well initially, which means that the model leads to substantial errors. By adjusting the SABR model with the method described in this paper, a reasonable fit was obtained. The benefit of using the simpler approach is that smaller companies, with fewer resources than larger companies, can easily program and implement this model.

There are still open questions on the topic of estimating volatility surfaces in an illiquid market. This research assignment implemented two models to capture the essence of volatility of volatility in the illiquid South African market. The research on this topic can be extended to other countries to test whether these two models hold for any illiquid market. This will be able to lead to a model which can be used in any smaller, illiquid market. Using the proposed models can also lead to further research the valuation of options with put-call parity. This will enable an investor to price options, which captures the fact that volatility is volatile in itself, more accurately.

To conclude this research assignment, taking into account that smaller organisations has arguably fewer resources, the author is led to believe that the Quadratic Deterministic approach leads to more accurate results and is therefore preferred above the SABR model. It is therefore a suitable model to estimate volatility surfaces in the illiquid South African market.

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APPENDIX A:Matlab Code for the SABR model

```
%Step 1: Load market data, vector of market volatilities, vector of
market strikes, settlement date (‘day, month, year), vector of
exercise dates

>>YearsToExercise = yearfrac(Settle, ExerciseDates, 1)

>>NumMaturities = length(YearsToExercise)

%Step 2: Calibrate SABR parameters for each maturity

>> Beta = 0.7;

Betas = repmat(Beta, NumMaturities, 1);

Alphas = zeros(NumMaturities, 1);

Rhos = zeros(NumMaturities, 1);

Nus = zeros(NumMaturities, 1);

>>options = optimoptions('lsqnonlin','Display','none');

>>for k = 1:NumMaturities

    % This function solves the SABR at-the-money volatility equation
as a

    % polynomial of Alpha

alpharoots = @(Rho,Nu) roots([...

        (1 - Beta)^2*YearsToExercise(k)/24/CurrentForwardValues(k)^(2
- 2*Beta) ...

        Rho*Beta*Nu*YearsToExercise(k)/4/CurrentForwardValues(k)^(1 -
Beta) ...

        (1 + (2 - 3*Rho^2)*Nu^2*YearsToExercise(k)/24) ...

        -ATMVolatilities(k)*CurrentForwardValues(k)^(1 - Beta)]);


  % This function converts at-the-money volatility into Alpha by
  picking the smallest positive real root,    % as proposed by West
  (2005).

atmVol2SabrAlpha = @(Rho,Nu) min(real(arrayfun(@(x) ...

        x*(x>0) + realmax*(x<0 || abs(imag(x))>1e-6),
alpharoots(Rho,Nu))));


    % Fit Rho and Nu (while converting at-the-money volatility into
Alpha)

objFun = @(X) MarketVolatilities(:,k) - ...
```

```
%blackvolbysabr code can be found at the end of this Appendix
blackvolbysabr(atmVol2SabrAlpha(X(1), X(2)), ...
        Beta, X(1), X(2), Settle, ExerciseDates(k),
CurrentForwardValues(k), ...
MarketStrikes(:,k));


    X = lsqnonlin(objFun, [0 0.5], [-1 0], [1 Inf], options);
    Rho = X(1);
    Nu = X(2);
% Get final Alpha from the calibrated parameters
    Alpha = atmVol2SabrAlpha(Rho, Nu);


Alphas(k) = Alpha;
Rhos(k) = Rho;
Nus(k) = Nu;
end
>>CalibratedPrameters = array2table([Alphas Betas Rhos Nus],...
    'VariableNames',{'Alpha' 'Beta' 'Rho' 'Nu'},...
    'RowNames',{'Expiry date 1';'Expiry date 2';...
    'Expiry date 3';'Expiry date 4'})
%Step 3: Construct volatility surface
%Create any vector of strikes
>>ComputedVols = zeros(length(PlottingStrikes), NumMaturities)
>>for k = 1:NumMaturities
ComputedVols(:,k) = blackvolbysabr(Alphas(k), Betas(k), Rhos(k),
Nus(k), Settle, ...
ExerciseDates(k), CurrentForwardValues(k), PlottingStrikes);
end
figure;
surf(YearsToExercise, PlottingStrikes, ComputedVols);
xlim([0 1]); ylim([290 410]); zlim([0.15 0.3]);
view(113,32);
```

```
set(gca, 'Position', [0.13 0.11 0.775 0.815], ...
    'PlotBoxAspectRatioMode', 'manual');
xlabel('Years to exercise', 'Fontweight', 'bold');
ylabel('Strike', 'Fontweight', 'bold');
zlabel('Implied SABR Black volatility', 'Fontweight', 'bold')
```

%Below is the blackvolbysabr function used in the algorithm above

```
Blackvolbysabrfunction outVol = blackvolbysabr(Alpha, Beta, Rho, Nu, Settle, ExerciseDate, ForwardValue, Strike, varargin)
%% Input argument checking
 % Number of inputs must be >=8 and <=12.
narginchk(8, 12);
if (~isnumeric(Alpha))||(~isscalar(Alpha))||(Alpha<=0)
error(message('fininst:blackvolbysabr:invalidAlpha'));
end
if (~isnumeric(Nu))||(~isscalar(Nu))||(Nu<=0)
error(message('fininst:blackvolbysabr:invalidNu'));
end
if (~isnumeric(Beta))||(~isscalar(Beta))||(Beta<0)||(Beta>1)
error(message('fininst:blackvolbysabr:invalidBeta'));
end

if (~isnumeric(Rho))||(~isscalar(Rho))||(Rho<-1)||(Rho>1)
error(message('fininst:blackvolbysabr:invalidRho'));
end
if
(~isnumeric(ForwardValue))||(~isvector(ForwardValue))||any(ForwardValu
e<=0)
error(message('fininst:blackvolbysabr:invalidForwardValue'));
end
if (~isnumeric(Strike))||(~isvector(Strike))||any(Strike<=0)
error(message('fininst:blackvolbysabr:invalidStrike'));
```

```
end
try
    Settle = datenum(Settle);
ExerciseDate = datenum(ExerciseDate);
catch ME
newMsg = message('fininst:blackvolbysabr:invalidSettleExerciseDate');
newME = MException(newMsg.Identifier,getString(newMsg));
newME = addCause(newME,ME);
throw(newME)
end
if (~isscalar(Settle))||(~isscalar(ExerciseDate))
error(message('fininst:blackvolbysabr:invalidSettleExerciseDate'));
end
if (Settle >ExerciseDate)
error(message('fininst:blackvolbysabr:ExerciseDateBeforeSettle'));
end
 p = inputParser;
p.addParameter('Basis',0,@(x) isvalidbasis(x)&&isscalar(x));
p.addParameter('Model','Hagan2002',@(x)
any(validatestring(lower(x),{'hagan2002','obloj2008'})));
try
p.parse(varargin{:});
catch ME
newMsg = message('fininst:blackvolbysabr:invalidInputs');
newME = MException(newMsg.Identifier,getString(newMsg));
newME = addCause(newME,ME);
throw(newME)
end
 Basis = p.Results.Basis;
Model = p.Results.Model;
 Time = yearfrac(Settle, ExerciseDate, Basis);
 [ForwardValue,Strike] = finargsz(1,ForwardValue(:),Strike(:));
```

```
NumVols = length(Strike);

outVol = zeros(NumVols,1);

 % Special case: At-the-money, (Forward == Strike);

ATMidx = find(abs(ForwardValue - Strike) <= eps(max(ForwardValue, Strike)));

V1 = ForwardValue(ATMidx).^(1 - Beta);

V2 = (1 - Beta).^2.*Alpha.^2./24./V1.^2 + ...

    0.25.*Rho.*Beta.*Nu.*Alpha./V1 + ...

    (2 - 3.*Rho.^2).*Nu.^2./24;

outVol(ATMidx) = Alpha.*(1 + V2.*Time)./V1;

NATMidx = setdiff((1:NumVols)',ATMidx); % Get non-ATM indices


% Special case: Beta == 1

if ((1 - Beta) <= eps(max(1, Beta)))

    z = Nu./Alpha.*log(ForwardValue(NATMidx)./Strike(NATMidx));

    x = log((sqrt(1 - 2.*Rho.*z + z.^2) + z - Rho)./(1 - Rho));

    V2 = 0.25.*Rho.*Alpha.*Nu + (2 - 3.*Rho.^2).*Nu.^2./24;

outVol(NATMidx) = Alpha.*z.*(1 + V2.*Time)./x;


% General case:

else

switch lower(Model)

case 'hagan2002' % Original model by Hagan(2002)

            V1 = (ForwardValue(NATMidx).*Strike(NATMidx)).^((1 - Beta)/2);

            z = Nu./Alpha.*V1.*log(ForwardValue(NATMidx)./Strike(NATMidx));

            x = log((sqrt(1 - 2.*Rho.*z + z.^2) + z - Rho)./(1 - Rho));

            V2 = (1 - Beta).^2.*Alpha.^2./24./V1.^2 + ...

                0.25.*Rho.*Beta.*Nu.*Alpha./V1 + ...

                (2 - 3.*Rho.^2).*Nu.^2./24;
```

```
            V3 = (1 - 
Beta).^2.*(log(ForwardValue(NATMidx)./Strike(NATMidx))).^2./24 + ...
                (1 - 
Beta).^4.*(log(ForwardValue(NATMidx)./Strike(NATMidx))).^4./1920;
outVol(NATMidx) = Alpha.*z.*(1 + V2.*Time)./x./V1./(1 + V3);
case 'obloj2008' % Version by Obloj(2008)
            z = Nu.*(ForwardValue(NATMidx).^(1 - Beta) - ...
Strike(NATMidx).^(1 - Beta))./Alpha./(1 - Beta);
            x = log((sqrt(1 - 2.*Rho.*z + z.^2) + z - Rho)./(1 - 
Rho));
            V1 = (ForwardValue(NATMidx).*Strike(NATMidx)).^((1 - 
Beta)/2);
            V2 = (1 - Beta).^2.*Alpha.^2./24./V1.^2 + ...
                0.25.*Rho.*Beta.*Nu.*Alpha./V1 + ...
                (2 - 3.*Rho.^2).*Nu.^2./24;
outVol(NATMidx) = Nu.*log(ForwardValue(NATMidx)./Strike(NATMidx)).*(1 
+ V2.*Time)./x;
end
end % end of if
```

APPENDIX B: R code Quadratic Deterministic Model

Set directory of Path to file

```
>setwd("Enter path of cleaned data on hard drive")


#Function to optimise coefficients for data
Optimise<-function(Data)
#Data must be in csv format
#Input is in string format e.g. "Sept14"
#Outputs are the coefficients
{
  Date<-paste(Data,".csv",sep="")
ExpDate<-read.csv(Date,header=TRUE)
  data1<-as.data.frame(ExpDate)
  M1<-data1$M
  Vol1<-data1$Vol
  MS1<-M1^2
  Quad1<-lm(Vol1~M1+MS1)
summary(Quad1)
  b01<-summary(Quad1)$coefficients[1,1]
  b11<-summary(Quad1)$coefficients[2,1]
  b21<-summary(Quad1)$coefficients[3,1]
Coeff<-list(b0=b01,b1=b11,b2=b21)
}



##############################################################3


ThetaLambda<-function(Date1,Date2,Date3,Date4,Current,Expiry)
#Current and Expiry inputs must is as follows year-month-day in string
format e.g. "2007-05-22"
#Expiry is a vecotr of all the expiry dates (of size 4) e.g.
c("","","","")
```

```
{
  Expiry<-as.Date(Expiry)
  Current<-as.Date(Current)
  t1<-(Expiry[1]-Current)*12/365
  t2<-(Expiry[2]-Current)*12/365
  t3<-(Expiry[3]-Current)*12/365
  t4<-(Expiry[4]-Current)*12/365

  dat0<-
data.frame(b0=c(Mar14$b0,Jun14$b0,Sept14$b0,Dec14$b0),t=c(t1,t2,t3,t4)
)
  min.RSS0 <- function(data, par) { with(data, sum((b0 -
par[1]/(t^par[2]))^2))}
  result0<- optim(par = c(0, 0), min.RSS0, data = dat0)

  dat1<-
data.frame(b1=c(Mar14$b1,Jun14$b1,Sept14$b1,Dec14$b1),t=c(t1,t2,t3,t4)
)
  min.RSS1 <- function(data, par) { with(data, sum((b1 -
par[1]/(t^par[2]))^2))}
  result1<- optim(par = c(0, 0), min.RSS1, data = dat1)

  dat2<-
data.frame(b2=c(Mar14$b2,Jun14$b2,Sept14$b2,Dec14$b2),t=c(t1,t2,t3,t4)
)
  min.RSS2 <- function(data, par) { with(data, sum((b2 -
par[1]/(t^par[2]))^2))}
  result2 <- optim(par = c(0, 0), min.RSS2, data = dat2)


  theta0<-result0$par[1]
  lambda0<-result0$par[2]
  theta1<-result1$par[1]
  lambda1<-result1$par[2]
  theta2<-result2$par[1]
```

```
  lambda2<-result2$par[2]

output<-list(t0=theta0, l0=lambda0,
t1=theta1,l1=lambda1,t2=theta2,l2=lambda2)

}

######################################################################
#

Mar14<-Optimise("Mar14")

Jun14<-Optimise("Jun14")

Sept14<-Optimise("Sept14")

Dec14<-Optimise("Dec14")

Expiry<-c("2014-03-20","2014-06-19","2014-09-18","2014-12-18")

outputs<-ThetaLambda(Mar14,Jun14,Sept14,Dec14,"2014-03-19",Expiry)

######################################################################
##
```

APPENDIX C: Screenshot of raw ALSI data

Safex Equity Derivatives Market Statistics for Thursday 19 December 2013

Interest on Initial Margin : 4.8700%

Total Margin on Deposit : R 13,389,580,433.50

	Contract	ExpiryDate		Strike Pri	Spot Pric	Vol	Bid	Offer	M-t-M	First	Last	High	Low	Deals	Conts	Value R	Open Int	Delta
68	ALSI	20/03/2014	C	32000.00	39607	24	39742	39759	78069	0	0	0	0	0	0	0	1000	0.9706
69	ALSI	20/03/2014	C	33400.00	39607	22	39742	39759	64577	0	0	0	0	0	0	0	18	0.9459
70	ALSI	20/03/2014	C	35000.00	39607	21	39742	39759	49658	0	0	0	0	0	0	0	1000	0.8968
71	ALSI	20/03/2014	C	36700.00	39607	20	39742	39759	34875	0	0	0	0	0	0	0	20	0.8065
72	ALSI	20/03/2014	C	37000.00	39607	19	39742	39759	32438	0	0	0	0	0	0	0	500	0.7853
73	ALSI	20/03/2014	C	37700.00	39607	19	39742	39759	26992	0	0	0	0	0	0	0	13	0.7296
74	ALSI	20/03/2014	C	38000.00	39607	19	39742	39759	24786	0	0	0	0	0	0	0	7000	0.7028
75	ALSI	20/03/2014	C	38500.00	39607	18	39742	39759	21296	0	0	0	0	0	0	0	100	0.6545
76	ALSI	20/03/2014	C	38900.00	39607	18	39742	39759	18673	0	0	0	0	0	0	0	44	0.6129
77	ALSI	20/03/2014	C	39000.00	39607	18	39742	39759	18039	0	0	0	0	0	0	0	51	0.6022
78	ALSI	20/03/2014	C	39050.00	39607	18	39742	39759	17734	0	0	0	0	1	16	301	16	0.5967
79	ALSI	20/03/2014	C	39300.00	39607	18	39742	39759	16216	0	0	0	0	0	0	0	380	0.5690
80	ALSI	20/03/2014	C	39800.00	39607	17	39742	39759	13396	0	0	0	0	0	0	0	63	0.5114
81	ALSI	20/03/2014	C	39900.00	39607	17	39742	39759	12869	0	0	0	0	0	0	0	125	0.4997
82	ALSI	20/03/2014	C	40000.00	39607	17	39742	39759	12354	0	0	0	0	0	0	0	2500	0.4878
83	ALSI	20/03/2014	C	40650.00	39607	17	39742	39759	9290	0	0	0	0	0	0	0	51	0.4098
84	ALSI	20/03/2014	C	40950.00	39607	16	39742	39759	8056	0	0	0	0	0	0	0	44	0.3739
85	ALSI	20/03/2014	C	41000.00	39607	16	39742	39759	7856	0	0	0	0	0	0	0	14	0.3679
86	ALSI	20/03/2014	C	41100.00	39607	16	39742	39759	7472	0	0	0	0	1	16	128	16	0.3560
87	ALSI	20/03/2014	C	41450.00	39607	16	39742	39759	6225	0	0	0	0	0	0	0	184	0.3151
88	ALSI	20/03/2014	C	41650.00	39607	16	39742	39759	5576	0	0	0	0	0	0	0	63	0.2922
89	ALSI	20/03/2014	C	41700.00	39607	16	39742	39759	5417	0	0	0	0	0	0	0	7	0.2865
90	ALSI	20/03/2014	C	41900.00	39607	16	39742	39759	4832	0	0	0	0	0	0	0	131	0.2644
91	ALSI	20/03/2014	C	42000.00	39607	16	39742	39759	4559	0	0	0	0	0	0	0	1780	0.2537
92	ALSI	20/03/2014	C	42050.00	39607	16	39742	39759	4426	0	0	0	0	0	0	0	512	0.2484
93	ALSI	20/03/2014	C	42100.00	39607	16	39742	39759	4291	0	0	0	0	0	0	0	300	0.2430
94	ALSI	20/03/2014	C	42300.00	39607	16	39742	39759	3799	0	0	0	0	0	0	0	370	0.2225
95	ALSI	20/03/2014	C	42350.00	39607	15	39742	39759	3682	0	0	0	0	0	0	0	100	0.2175
96	ALSI	20/03/2014	C	42450.00	39607	15	39742	39759	3451	0	0	0	0	0	0	0	13	0.2075
97	ALSI	20/03/2014	C	42500.00	39607	15	39742	39759	3342	0	0	0	0	0	0	0	49	0.2026
98	ALSI	20/03/2014	C	42850.00	39607	15	39742	39759	2643	0	0	0	0	0	0	0	49	0.1701
99	ALSI	20/03/2014	C	43000.00	39607	15	39742	39759	2376	0	0	0	0	0	0	0	4200	0.1569
100	ALSI	20/03/2014	C	43400.00	39607	15	39742	39759	1766	0	0	0	0	0	0	0	267	0.1249
101	ALSI	20/03/2014	C	44000.00	39607	14	39742	39759	1085	0	0	0	0	0	0	0	500	0.0850
102	ALSI	20/03/2014	C	44200.00	39607	14	39742	39759	914	0	0	0	0	0	0	0	320	0.0740
103	ALSI	20/03/2014	C	45400.00	39607	14	39742	39759	285	0	0	0	0	0	0	0	43	0.0280

Safex Equity Derivatives Market Statistics for Wednesday 19 March 2014

Interest on Initial Margin : 5.3200%

Total Margin on Deposit : R 17,802,139,013.00

	Contract	ExpiryDate		Strike Pri	Spot Pric	Vol	Bid	Offer	M-t-M	First	Last	High	Low	Deals	Conts	Value R	Open Int	Delta
151	ALSI	20/03/2014	C	32000.00	42007	33.14	41982	41993	99880	0	0	0	0	0	0	0	1000	1.0000
152	ALSI	20/03/2014	C	33400.00	42007	30.89	41982	41993	85880	0	0	0	0	0	0	0	18	1.0000
153	ALSI	20/03/2014	C	35000.00	42007	28.45	41982	41993	69880	0	0	0	0	0	0	0	1000	1.0000
154	ALSI	20/03/2014	C	36700.00	42007	25.87	41982	41993	52880	0	0	0	0	0	0	0	20	1.0000
155	ALSI	20/03/2014	C	37000.00	42007	25.42	41982	41993	49880	0	0	0	0	0	0	0	500	1.0000
156	ALSI	20/03/2014	C	37700.00	42007	24.36	41982	41993	42880	0	0	0	0	0	0	0	13	1.0000
157	ALSI	20/03/2014	C	38000.00	42007	23.92	41982	41993	39880	0	0	0	0	0	0	0	7000	1.0000
158	ALSI	20/03/2014	C	38500.00	42007	23.20	41982	41993	34880	0	0	0	0	0	0	0	100	1.0000
159	ALSI	20/03/2014	C	38900.00	42007	22.62	41982	41993	30880	0	0	0	0	0	0	0	44	1.0000
160	ALSI	20/03/2014	C	39050.00	42007	22.41	41982	41993	29380	0	0	0	0	0	0	0	16	1.0000
161	ALSI	20/03/2014	C	39300.00	42007	22.04	41982	41993	26880	0	0	0	0	0	0	0	380	1.0000
162	ALSI	20/03/2014	C	40000.00	42007	21.04	41982	41993	19880	0	0	0	0	0	0	0	2500	1.0000
163	ALSI	20/03/2014	C	40400.00	42007	20.48	41982	41993	15880	0	0	0	0	0	0	0	2	0.9998
164	ALSI	20/03/2014	C	40950.00	42007	19.72	41982	41993	10391	0	0	0	0	0	0	0	44	0.9925
165	ALSI	20/03/2014	C	41000.00	42007	19.65	41982	41993	9895	0	0	0	0	0	0	0	29	0.9898
166	ALSI	20/03/2014	C	41100.00	42007	19.51	41982	41993	8908	0	0	0	0	0	0	0	16	0.9821
167	ALSI	20/03/2014	C	41200.00	42007	19.37	41982	41993	7931	0	0	0	0	0	0	0	1000	0.9695
168	ALSI	20/03/2014	C	41450.00	42007	19.02	41982	41993	5571	0	0	0	0	0	0	0	102	0.9033
169	ALSI	20/03/2014	C	41500.00	42007	18.95	41982	41993	5123	0	0	0	0	0	0	0	110	0.8817
170	ALSI	20/03/2014	C	41700.00	42007	18.67	41982	41993	3460	0	0	0	0	0	0	0	7	0.7609
171	ALSI	20/03/2014	C	41900.00	42007	18.39	41982	41993	2089	0	0	0	0	0	0	0	131	0.5881
172	ALSI	20/03/2014	C	42000.00	42007	18.25	41982	41993	1541	0	0	0	0	0	0	0	2532	0.4900
173	ALSI	20/03/2014	C	42050.00	42007	18.19	41982	41993	1305	0	0	0	0	0	0	0	512	0.4403
174	ALSI	20/03/2014	C	42100.00	42007	18.12	41982	41993	1093	0	0	0	0	0	0	0	300	0.3912
175	ALSI	20/03/2014	C	42200.00	42007	17.98	41982	41993	741	0	0	0	0	0	0	0	75	0.2979
176	ALSI	20/03/2014	C	42250.00	42007	17.92	41982	41993	600	0	0	0	0	0	0	0	500	0.2551
177	ALSI	20/03/2014	C	42300.00	42007	17.85	41982	41993	480	0	0	0	0	0	0	0	378	0.2154
178	ALSI	20/03/2014	C	42350.00	42007	17.78	41982	41993	378	249	249	0	0	2	500	157	350	0.1794
179	ALSI	20/03/2014	C	42450.00	42007	17.65	41982	41993	226	0	0	0	0	0	0	0	13	0.1190
180	ALSI	20/03/2014	C	42500.00	42007	17.58	41982	41993	171	1700	1700	1700	1700	2	6	10	59	0.0947
181	ALSI	20/03/2014	C	42750.00	42007	17.25	41982	41993	33	0	0	0	0	0	0	0	5	0.0234
182	ALSI	20/03/2014	C	42850.00	42007	17.11	41982	41993	15	0	0	0	0	0	0	0	49	0.0118
183	ALSI	20/03/2014	C	43000.00	42007	16.91	41982	41993	4	0	0	0	0	0	0	0	3100	0.0036
184	ALSI	20/03/2014	C	43200.00	42007	16.64	41982	41993	1	0	0	0	0	0	0	0	20	0.0006
185	ALSI	20/03/2014	C	43300.00	42007	16.51	41982	41993	0	0	0	0	0	0	0	0	400	0.0002
186	ALSI	20/03/2014	C	43350.00	42007	16.44	41982	41993	0	0	0	0	0	0	0	0	400	0.0001