Applying the Nelson & Siegel model to the South African zero coupon yield curve

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Abstract

The importance of the term structure of interest rates to financial market participants has led to the development of mathematical models to explain the shape of the yield curve. One such model that is considered in this paper is the Nelson & Siegel (1987) three factor model. This model expresses the entire term structure of interest rates by three latent factors. Determining these factors for the South African zero coupon yield curve over time could prove useful for many purposes. This paper firstly evaluates the fit of this model to the considered term structure of interest rates. Secondly, the latent factors are extracted over time in order to forecast the zero rates. Additionally, an investigation of whether combining these factors with macroeconomic variables to provide meaningful forecasts is made. Finally, the application of the Nelson & Siegel (1987) model for purpose of extrapolating the term structure of interest rates is critically assessed. The simplicity of the Nelson & Siegel (1987) model compared to various other term structure models may perhaps result in it being applied by less quantitative literate market participants. It was found that the Nelson & Siegel (1987) model provided a good in sample fit for the South African zero coupon yield curve and a reasonable out of sample forecast. Additionally, for extrapolation purposes, the model performed adequately.

Opsomming

Die belangrikheid van die termynstruktuur van rentekoerse om deelnemers van finansiële markte het gelei tot die ontwikkeling van wiskundige modelle, om die vorm van die termynstruktuur te verduidelik. Een van hierdie modelle wat beskou word in hierdie projek is die Nelson & Siegel (1987) drie faktor model. Hierdie model gee uitdrukking aan die hele termynstruktuur van rentekoerse deur drie latente faktore. Die bepaling van hierdie faktore vir die Suid-Afrikaanse termynstruktuur met verloop van tyd kan nuttig wees vir 'n verskeidenheid van doeleindes. Hierdie projek probeer eerstens die pas van hierdie model te evalueer om die termynstruktuur van rentekoerse vas stel. Tweedens word die latente faktore ontrek oor tyd om die termynstruktuur te voorspel. Daarna word ‘n ondersoek ingestel om te sien of makro-ekonomiese veranderlikes by te voeg, ‘n beteknisvolle voorspelling sal maak. Ten slotte, is die toepassing van die Nelson & Siegel (1987) model vir doeleindes van ekstrapolasie van die termynstruktuur vir rentekoerse krities beoordeel. Die duidelikheid van die Nelson & Siegel (1987) model in vergelyking met verskeie ander termynstruktuur modelle kan dalk help dat dit deur minder kwantitatiewe mark deelnemers toegepas word. Daar is bevind dat die Nelson & Siegel (1987) model voorsien 'n goeie passing vir die streekproef wat geskik is vir die Suid-Afrikaanse termynstruktuur en 'n redelike uit streekproef vooruitskatting. Verder vir ekstrapolasie doeleindes het die model voldoende presteer.

Table of contents

Applying the Nelson & Siegel model to the South African zero coupon yield curve		**1**
PLAGIARISM DECLARATION		**2**
Acknowledgements		**3**
Abstract		**4**
Opsomming		**5**
Table of contents		**6**
List of tables		**8**
List of figures		**9**
List of abbreviations and/or acronyms		**10**
CHAPTER 1 INTRODUCTION		**11**
1.1	INTRODUCTION	11
1.2	PROBLEM STATEMENT	11
1.3	RESEARCH QUESTION	12
1.4	RESEARCH OBJECTIVES	12
1.5	RESEARCH BENEFITS/IMPORTANTS	12
1.6	RESEARCH DESIGN / CHAPTER OVERVIEW	12
CHAPTER 2 LITERATURE REVIEW		**14**
2.1	INTRODUCTION	14
2.2	YIELD CURVE BACKGROUND	14
2.3	LINK BETWEEN BOND PRICES, SPOT AND FORWARD RATE.	15
2.4	THE MODEL	16
2.4.1	Factors affecting the yield curve	17
2.4.2	Fitting and forecasting the yield curve using time varying latent factors	18
2.4.3	Inclusion of macroeconomic variables	19
2.4.4	Accounting for changes in the monetary policy regimes	21
2.4.5	Yield curve extrapolation	22
2.4.6	Yield curve forecasting	23
2.5	SUMMARY	23
CHAPTER 3 30 YEAR YIELD CURVE MODELLING AND FORECASTING		**24**
3.1	INTRODUCTION	24
3.2	DATA DESCRIPTION AND METHODOLOGY	24
3.2.1	Data Description	24
3.2.2	Methodology	26
3.3	RESULTS	31
3.3.1	Estimation with fixed λ	31
3.3.2	Estimation with varying λ	36

3.3.3 Relationship with macro variables 38

3.3.4 Forecasting results 39

3.4 SUMMARY 42

CHAPTER 4 10 YEAR ZERO YIELD CURVE EXTRAPOLATION 43

4.1 INTRODUCTION 43

4.2 METHODOLOGY 43

4.3 RESULTS 46

4.4 SUMMARY 51

CHAPTER 5 CONCLUSION AND OPENING QUESTIONS 52

REFERENCES 54

List of tables

Table 3.1: λ estimation results	32
Table 3.2: Summary statistics estimated factors	33
Table 3.3: VAR(1) parameter estimation results	39
Table 3.4: forecasting for 2008:01 – 2008:12	39
Table 3.5: Forecast for 2012:01 – 2012:12	40
Table 3.6: Forecast for 2015:07 – 2016:06	41
Table 4.1: Fitted and extrapolated squared residuals	47

List of figures

Figure 2.1: βo $\beta 1$ and $\beta 2$ loading factors. 17
Figure 2.2: Effects of level, slope, and curvature on yield curve. 18
Figure 3.1: 30 year BEASSA yield curve 25
Figure 3.2: yield curves for altered decay factors λ. 28
Figure 3.3: loading factors for λ=0.0609 32
Figure 3.4: Estimated factors vs. approximated factors 33
Figure 3.5: Average yield curve 2004:02-2016:06 34
Figure 3.6: 30 year yield curve residuals 35
Figure 3.7: 30 year yield curve fit. 35
Figure 3.8: 30 yield curve residuals with varying λ 36
Figure 3.9: 30 yield curve plots 37
Figure 3.10: Squared residuals over time 37
Figure 4.1: Fitted vs extrapolated residuals. 44
Figure 4.2: level vs 360 month spot rate 46
Figure 4.3: 30 September 2004 extrapolated yield curve. 47
Figure 4.4: 31 January 2005 extrapolated yield curve. 49
Figure 4.5: 28 February 2007 extrapolated yield curve. 50
Figure 4.6: 30 March 2012 extrapolated yield curve. 51

List of abbreviations and/or acronyms

AR(1) – First order autoregressive process

BEASSA – Bond Exchange Actuarial Society South Africa

DNS- Dynamic Nelson Siegel

GRG- Generalised Reduced Gradient

JSE- Johannesburg Stock Exchange

RMSE- Root mean squared error

VAR(1)- First order vector autoregressive process

CHAPTER 1
INTRODUCTION

1.1 INTRODUCTION

"If you invest in equities, you should keep an eye on the bond market. If you invest in real estate you should keep an eye on the bond market. If you invest in bonds you should definitely keep an eye on the bond market" – Barry Nielson

The above statement highlights the importance that the bond market plays in all areas of the investment world. No other single illustration provides a comprehensive overview of the bond market as the yield curve, also referred to as the term structure of interest rates. Frabozzi (2005:139) defines the yield curve as a graphical representation of the relationship between fixed income securities of the same credit quality for different maturities. The yield curve is generally constructed from liquid traded fixed income securities and provides market participants with endless utility such as the setting of interest rates in order to provide future expectations about the market.

Given the importance of the yield curve, it is not surprising that mathematicians and statisticians have attempted to describe its behaviour in an analytical manner. For example, the model suggested by Nelson & Siegel (1987) describes the entire term structure of interest rates by a few simple mathematical functions. Since the publication of the paper by Nelson & Siegel (1987), many extensions and adjustments has been proposed and thus a detailed literature review is necessary.

According to the Johannesburg Stock Exchange (2016), R25 billion worth of trading occurs daily in the South African debt market therefore, there is no doubt that the yield curve plays an important role in the financial industry. Consequently, a study on the use of the Nelson & Siegel (1987) model could prove valuable for all market participants. There will be an examination of the use of the Nelson & Siegel (1987) term structure model for purposes of forecasting and extrapolating the yield curve, particularly for the South African debt market.

1.2 PROBLEM STATEMENT

The ideal situation would be a parsimonious model that is able to explain the behaviour of the term structure of interest rates as accurately as possible over time. This, however, is not the case as the term structure of interest rates is governed by many observable and unobservable factors, some of which are not quantifiable. Therefore, there exist a trade-off between accuracy and simplicity of the model. Thus, a thorough examination of the Nelson & Siegel (1987) model's applicability to the South African debt market is required.

1.3 RESEARCH QUESTION

Within the research paper, the question of whether the Nelson & Siegel (1987) model is sufficient to describe the behaviour of the South African term structure of interest rates and its degree of applicability for extrapolation and forecasting of the term structure of interest rates will be investigated.

1.4 RESEARCH OBJECTIVES

First and foremost, a thorough literature review is required to critically examine the Nelson & Siegel (1987) model as well as its adjustments and applications. Secondly, the practicality of Nelson & Siegel (1987) model for extrapolation and forecasting of the South African term structure of interest rates will be investigated.

1.5 RESEARCH BENEFITS/IMPORTANTS

This study will add to the limited available research of the South African yield curve construction. Also, inclusion of macroeconomic variables might increase the accuracy of modelling, forecasting and extrapolation of the South African yield curve. Thirdly, many financial and economic models depend on the zero yield curve. Finally, this will lead to the further development of the financial markets in South Africa.

1.6 RESEARCH DESIGN / CHAPTER OVERVIEW

This paper has two main objectives: firstly, assessing the applicability of the Nelson & Siegel (1987) model for modelling the term structure of interest rates dynamically over time. This dynamic behaviour might possibly interact with certain macroeconomic variables and this could possibly be used to forecast the entire term structure of interest rates. Secondly, an exploration of the usage of the Nelson & Siegel (1987) model to extend the term structure of interest rates for unobserved maturities will be investigated.

Chapter 1 gives a brief introduction to the framework and scope of this paper. Chapter 2 will provide a thorough discussion of the Nelson & Siegel (1987) model and its quantitative development. Additionally, specific applications of this model will be reviewed as well as the value added to field of term structure modelling. Chapter 3 firstly describes the methodology applied and the data used for the modelling and forecasting of the term structure of interest rates. Secondly, results obtained by applying this methodology will be presented and reviewed. Chapter 4 starts off by describing the methodology used to extrapolate the term structure of interest rates. Afterwards, the results obtained through the application of the methodology will be reviewed. Finally, Chapter 5 will conclude the results achieved in this paper given what it was set out to accomplish. Additionally, the limitations of this research will be evaluated and propositions will be made for possible future research.

CHAPTER 2
LITERATURE REVIEW

2.1 INTRODUCTION

In this chapter, a review of relevant literature concerning the modelling of the yield curve is discussed in detail. Initially, some background regarding the construction of the yield curve with the appropriate theory explaining the possible observed yield curve shapes will be given. Subsequently, the considered model used to estimate the shape of the yield curve is reviewed and the historical applications of this model are studied. Effectively, the literature review delves into the theoretical concepts as well as possible methods to implement the theory needed to model a yield curve.

2.2 YIELD CURVE BACKGROUND

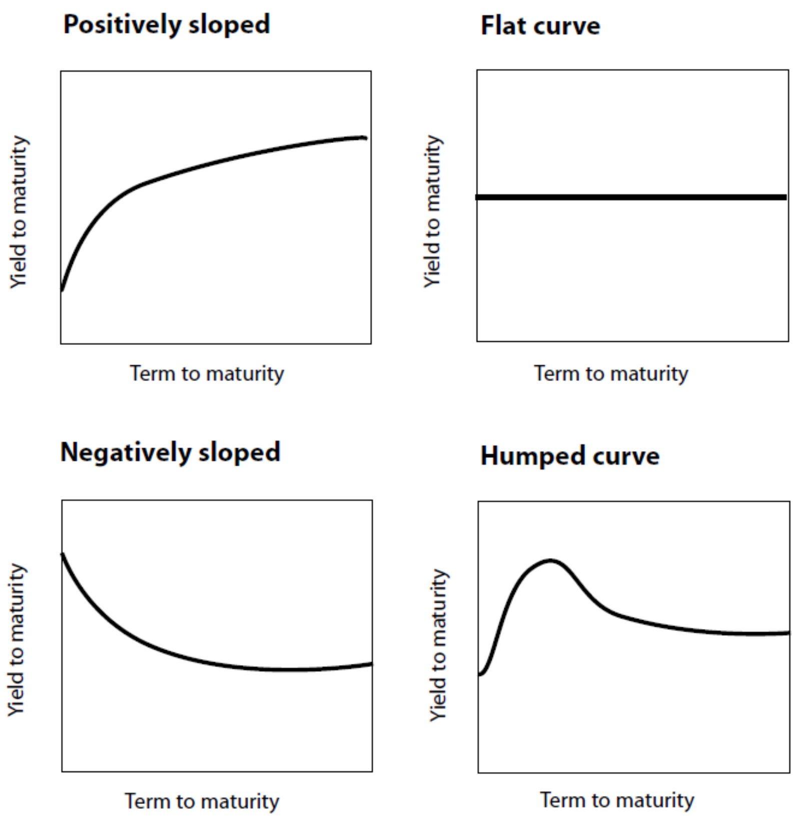
Fabozzi (2005:940) defines a zero coupon bond as an issued debt security that provides the holder of the bond a payoff equal to the face value of the bond at some future date called the maturity date. The interest or yield earned on this bond is known as the spot or zero rate. Fabozzi (2005:142) also states that in practice, zero coupon bonds only exist for short term maturities, i.e. up to 1 year, and therefore a method known as bootstrapping is used to extract the zero rates from coupon bearing bonds with longer maturities. A graphical representation of the zero rates at varying maturities is known as the theoretical zero coupon yield curve.

Furthermore, Stander (2005:2) explains that there are four distinct possible yield curve shapes (illustrated in Figure 1.1) referred to as positively sloped, negatively sloped, flat and humped curves.

The respective shapes shown in Figure 1.1 are, as explained by Stander (2005:3), determined by the following 3 theories:

1. Market segmentation theory: succinctly, this theory states that the zero rates at each maturity date are determined by the supply and demand subject to the obligations that large financial institutions have in a certain maturity bucket.
2. Pure expectations theory: this theory is conditional on the assumption that all market participants are indifferent to the term of an investment and only care about achieving the highest possible return over a given investment period.
3. Liquidity preference theory: this theory essentially drops the assumption of the pure expectations theory by assuming that market participants do care about the investment term, and therefore, require a premium for holding longer term illiquid assets. Note that this theory implies that only an upward sloping yield curve can exist.

Figure 1.1 Yield curve shapes.



Source: Stander (2005)

Now, because the relevant background theory of the term structure of interest rates was discussed, an applicable model will now be introduced.

2.3 LINK BETWEEN BOND PRICES, SPOT AND FORWARD RATE.

According to Diebold & Li (2006: 339-340), to fit a model to the yield curve, there must be an understanding of how the bond prices, the forward rates and the spot rates relate. The theoretical relationship is as follows:

$$p_t(\tau) = e^{-\tau y_t(\tau)} \tag{2.1}$$

$$f_t(\tau) = -\frac{p'_t(\tau)}{p_t(\tau)} \tag{2.2}$$

$$y_t(\tau) = \int_0^{\tau} f_t(\tau)\, d\tau \qquad (2.3)$$

Where:

- $f_t(\tau)$ is the instantaneous forward rate at maturity τ
- $y_t(\tau)$ is the yield to maturity (spot rate)

$p_t(\tau)$ is the price of R1 τ-period discount bond at time t (discount curve)

According to Diebold & Rudebusch (2013:16), to estimate the yield curve, the estimates of the discounts curves were modelled using polynomial spines and later on, exponential splines. This method, however, usually leads to a negative forward rates. Hence the alternative method was to rather use the Fama & Bliss (1987) method, which rather estimates the forward rates at different maturities to estimates the yield curve. The latter method is what this paper will used to estimate the yield curve.

A model that that can been used to model and forecast a yield curve will now be introduced-the Nelson & Siegel (1987) model. Its main attraction is its simplicity. This model revolutionised how the dynamics of yield curves are modelled and has been applied in numerous countries since its creation.

2.4 THE MODEL

Nelson & Siegel (1987:473) were the first to propose a simple parsimonious model for modelling the yield curve. The model was motivated by the need to represent the entire yield curve with just a few parameters. Previous observation of the shape of the yield curve and the expectation theory of the term structure of interest rates had suggested that the yield curve was either monotonic, humped or S shaped. Accordingly, Nelson & Siegel (1987) developed a model that produced these shapes. The proposed model was tested on the US Treasury Bills data and it explained most of the variation in the treasury bills.

The proposed modelled was as follows:

$$f_t(\tau) = \beta_o + \beta_1 \times e^{-\frac{\tau}{\lambda}} + \beta_2 \times \frac{\tau}{\lambda} \times e^{-\frac{\tau}{\lambda}} \qquad (2.4)$$

$$y_t(\tau) = \beta_o + (\beta_1 + \beta_2) \times \left(\frac{1 - e^{-\frac{\tau}{\lambda}}}{\frac{\tau}{\lambda}} \right) - \beta_2 \times e^{-\frac{\tau}{\lambda}} \qquad (2.5)$$

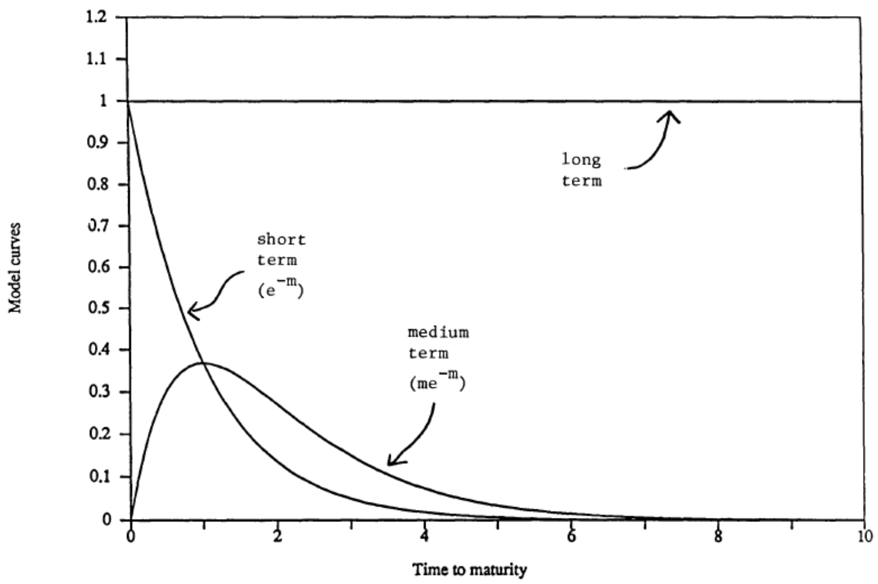
Where

- $f_t(\tau)$ is the instantaneous forward rate at maturity τ

- $y_t(\tau)$ is the yield to maturity (spot rate)
- $\beta_o, \beta_1, \beta_2$ are constants determined from initial conditions
- $\frac{1}{\lambda}$ is the decay factor

The implication of Nelson & Siegel (1987:476) model is that the coefficients $\beta_o, \beta_1, \beta_2$ determine respectively the extent of the contributions that the short, medium and the long term maturities make to the yield curve. Figure 2.1 graphically illustrates the components that load on each of the coefficients $\beta_o, \beta_1, \beta_2$. The long term component does not decay to zero as the maturity tends to infinity, but rather remains constant. On the other hand, as maturity tends to infinity, the short term component (which determines the rates in the short term) starts of at 1 and then proceeds to decay at an exponential rate until it reaches zero. Finally, the medium term component starts at zero, maximises at some medium point maturity, and then finally decays to zero as the maturity tends to infinity. Note that λ determines the point where this medium term component reaches its maximum and that the medium term component decays at a slower rate compared to the short term component.

Figure 2.1: β_o β_1 and β_2 loading factors.

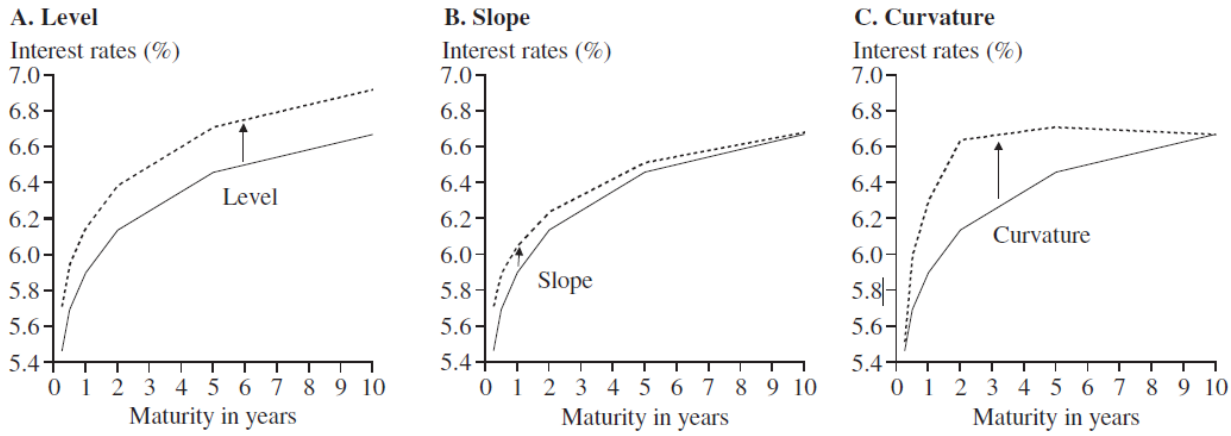


Source: Nelson & Siegel (1987)

2.4.1 Factors affecting the yield curve

To model a yield curve over time might at first seem complex because of its dynamism. However, Nelson & Siegel's (1987) recognition that the movements of a yield curve are governed by a few factors simplified the task. Thereafter to build on Nelson & Siegel (1987) research, Litterman & Scheinkman (1991:54, 57-58) investigated the common factors that affected the returns on US treasury fixed income securities in order to hedge a bond portfolio against interest rates movements. For this reason, the research used factor analysis and found that most of the variation in the bond returns were explained by three latent (or unobservable) factors of the yield curve. These were called the level, slope and curvature. Consequently, a 3 factor model was used. Furthermore, the analysis found that the level factor caused a parallel shift on the yield curve, the slope factor lowered the curve up to the 5 year maturity date and then raised it for longer maturity dates. Finally, the curvature factor affected the curvature of the yield curve. To substantiate, Figure 2.2 provides Wu's (2003:1) explanation of how a yield curve (solid line) responds to shocks (dashed line) in the level, slope and curvature factors:

Figure 2.2: Effects of level, slope, and curvature on yield curve.



Source: Wu (2003)

Notice that shocks in the level factor results in a parallel shift in the entire yield curve. However, shocks in the slope factor affect the short-end of the yield curve more than the other terms. Finally, shocks to the curvature factor affects the medium term part of the yield curve but barely affects the short- and long-end part of the yield curve.

Nevertheless, to model a dynamic yield curve using a static model like the Nelson & Siegel (1987) is not efficient. Instead, a dynamic model is required.

2.4.2 Fitting and forecasting the yield curve using time varying latent factors

According to Diebold & Rudebusch (2013:21), dynamic factor models are the preferred method of modelling yields because yield curve data can be represented more accurately when factor models are used, it is easier to statistically analyse factor models as opposed to non-factor models, and a great deal of financial economic theory makes use of factor models. Keeping this in mind, Diebold & Li (2006:337, 359-360) recognised that the 3 latent factors of the yield curve could in fact be interpreted as time varying: the level l_t, slope s_t and curvature c_t. First order autoregressive and vector autoregressive models, AR(1) and VAR(1), where fitted to these latent factors of the yield curve. Autoregressive models are used in statistics to describe processes that varying over time. The Diebold & Li (2006) then forecasted the yield curve by forecasting these 3 dynamic latent factors and compared this method to the other forecasting methods (for example the random walk). The research concluded that the 1 month ahead forecast using the Nelson & Siegel (1987) model for the yields with AR(1)/VAR(1) factor dynamics performed the same as the other forecasting methods, however, the 12 month ahead forecast using the Nelson & Siegel (1987) model for the yields with AR(1)/VAR(1) factor dynamics performed much better.

Note that, Diebold & Li (2006:340-341) slightly modified the original Nelson & Siegel (1987) model representation and redefined it as follows:

$$f_t(\tau) = l_t + s_t \times e^{-\lambda\tau} + c_t \times \lambda\tau \times e^{-\lambda\tau} \quad (2.6)$$

$$y_t(\tau) = l_t + s_t \times \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau}\right) + c_t \times \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) \quad (2.7)$$

Where

- $f_t(\tau)$ is the instantaneous forward rate at maturity τ
- $y_t(\tau)$ is the spot rate
- λ is the fixed decay factor
- l_t, s_t, c_t are time varying latent factors

The short term yields load more on s_t which is related to the slope, and the slope proxy is defined as $y_t(120) - y_t(3)$. The medium term yields load more on c_t which is related to the curvature, and the curvature proxy is defined as $2y_t(24) - y_t(3) - y_t(120)$. Finally, the long term yields load more on l_t which is related to the level, and the level proxy is defined as $y_t(120)$.

Up until this point, it is surprising that there had been no mention of macroeconomics despite the fundamental role that the yield curve plays in the economy. For example, Wu (2003:2) explains that adjusting the short-end of the yield curve to stabilise the economy (low inflation and maximum output target) is one tool that central banks use. In addition, since expected future short term rates directly affects the long term rates, macroeconomic variables directly influence the entire term structure of interest rates. Therefore, are dynamic unobservable factors alone sufficient to model a yield curve or can observable macroeconomic factors also be utilised to more accurately represent the dynamics of the yield curve?

2.4.3 Inclusion of macroeconomic variables

According to Diebold & Rudebusch (2013:22), previously, there was a disconnect between macro and finance literature. Finance literature assumed that short term rates were a function of a few latent factors and that the changes in the corresponding long term rates were determined by the changes in the risk premiums, which were also a function of these few latent factors. On the other hand, macro literature followed the expectation hypothesis. Gürkaynak & Wright (2012:332-333) explains that according to the expectation theory, the long term yields are only a function of the expected future short-rates. Therefore, if specific macroeconomic variables drive the short term rates set by the central bank, then the term structure of interest rates should consequently reflect the expected future value of the same macroeconomic variables. So in order to understand the underlying macroeconomic variables that drive the yield curve, a unification of these schools of thoughts is required.

Aruoba, Diebold & Rudebusch (2006: 310-311) noticed that there was no model that captured the bidirectional relationship between the latent dynamic factors of the yield curve and the macroeconomic variables. Previous models that attempted to include macroeconomic variables, like, for example, Ang & Piazzesi (2003), were only one directional: they assumed that the macroeconomic variables were independent of the yield curve. Below is an example of a unidirectional investigation of interactions of macroeconomic variables and the yield curve.

Morocco's sovereign debt market, although illiquid, is quickly developing. It also uses a pegged exchange rate (pegged to the US dollar and the Euro). Ahokpossi, Garcia-Martinez & Kemoe (2016) wanted to understand how effective the monetary and fiscal policy transmitted to the Moroccan yield curve. Ahokpossi *et al*. (2016) used the Diebold & Li (2006) model to estimate the latent factors of the Moroccan yield curve. Thereafter, they use the principal component analysis (PCA) to find an estimate for the various factors that represented the different sectors of the economy. Then they place the output of the PCA and the estimated latent factors in a VAR(1) model. Finally, they use an impulse response function to investigate the correlation between the macroeconomic variables and the latent factors. There was indeed an interaction.

Aruoba *et al.* (2006:312-313) went a step further and proposed a bidirectional model. A model of state space representation is used to describe the yield curve because the dynamic latent factors follow VAR(1). Below, are the transition equation for the latent factors and the measurement equation which links the yields, $y_t(\tau)$, to the dynamic latent factors:

$$\begin{pmatrix} l_t - \mu_l \\ s_t - \mu_s \\ c_t - \mu_c \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{13} \\ \vdots & \ddots & \vdots \\ a_{31} & \cdots & a_{33} \end{pmatrix} \times \begin{pmatrix} l_{t-1} - \mu_l \\ s_{t-1} - \mu_s \\ c_{t-1} - \mu_c \end{pmatrix} + \begin{pmatrix} \eta_t(l) \\ \eta_t(s) \\ \eta_t(c) \end{pmatrix} \tag{2.8}$$

$$\begin{pmatrix} y_t(\tau_1) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1 - e^{-\lambda\tau_1}}{\lambda\tau_1} & \frac{1 - e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1 - e^{-\lambda\tau_N}}{\lambda\tau_N} & \frac{1 - e^{-\lambda\tau_N}}{\lambda\tau_N} - e^{-\lambda\tau_N} \end{pmatrix} \times \begin{pmatrix} l_t \\ s_t \\ c_t \end{pmatrix} + \begin{pmatrix} \varepsilon_t(\tau_1) \\ \vdots \\ \varepsilon_t(\tau_N) \end{pmatrix} \tag{2.9}$$

$$
\begin{gathered}
(f_t - u) = A(f_{t-1} - u) + \eta_t \\
y_t = \Lambda f_t + \varepsilon_t \\
\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim WN\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix}\right]
\end{gathered} \tag{2.10}
$$

Where (2.10) is the state space system in matrix notation and:

- f_t is the vector containing time series of l_t, s_t, c_t factors
- u is the mean reverting value of f_t
- A parameter matrix for the transition equation
- Λ parameter matrix for the measurement equation
- $\eta_t \sim N(0, Q)$, i.i.d for $\mathrm{t} = 1, \ldots, \mathrm{T}$
- $\varepsilon_t \sim N(0, H)$, i.i.d for $\mathrm{t} = 1, \ldots, \mathrm{T}$
- Q is the transition equation error variance-covariance matrix
- H is the diagonal measurement error variance-covariance matrix

The relevant parameters, including the matrix Q and A, are estimated using the method of maximum likelihood. The above state space representation is then modified to incorporate the relevant macroeconomic variables by adding them to the vector f_t. Aruoba *et al.* (2006) extracted the latent dynamic factors and then related them to the following macroeconomic variables: inflation, capacity utilisation and the federal fund rate. Then the yield curve was forecasted. Thereafter, to investigate the correlations between the latent factors and the macroeconomic variables, an impulse response function and the variance decomposition were used. The findings were that a bidirectional relationship between the latent factors and the macroeconomic variables existed. However, there was a stronger relationship when looking at the effects of macroeconomic variables on the yield curve than the other way around. To substantiate, Alves, Cabral**,** Munclinge, Rodriguez & Waldo (2011) conducted the same study on Brazil's yield curve for the period 2004-2010 and the results were the same as Aruoba *et al.* (2006).

If it is accepted that central banks have some control over the term structure of interest rates through monetary policy instruments, how then can yield curves be modelled if there is a structural change in the monetary policy regime?

2.4.4 Accounting for changes in the monetary policy regimes

Levant & Ma (2016) performed the same analysis as Diebold & Li (2006) for the UK. The difference is that Levant & Ma (2016:127) accounted for structural changes in the monetary policy in the UK in 1992, when the policy moved from the exchange rate and monetary aggregate anchoring to inflation targeting. This implied that the whole analysis was performed for the period before and after the regime change, i.e. the sample period has to be divided in two parts. Doing this increased the accuracy of the latent factor estimations and consequently better describes their relationship with observable macroeconomic variables. Monetary policy interest rates, total industrial production and the inflation expectation were the chosen macroeconomic variables. According to Rudebusch & Svensson (1999) and Kozicki & Tinsley (2001), these are the minimum set of variables required to describe the economy. In addition, Levant & Ma (2016:119) use the diagonal Q matrix form as this has a negligible effect on parameter estimation and is easier to work with. The conclusion is that the yield curve volatility reduced post-monetary policy regime changes.

An emerging market example is Kaya (2013) that conducts a similar study for Turkey. Kaya (2013) investigated the macroeconomic and yield curve interactions by using the Dynamic Nelson-Siegel (DNS) framework for the periods: 1993-2002 (pre monetary policy changes) and then for 2002-2009 (after the monetary policy was changed to inflation targeting). The findings were that there was a change in the relationship between the macroeconomic variables and the latent factors of the yield curves after the structural break. More specifically, macroeconomic variables better explained the changes in the yield curve after the regime change. The implications are that the Central Bank has direct control in the interactions between the yield curve and the macroeconomic variables.

In the next section the various ways in which the yield curve can be extrapolated is discussed. Extrapolation of the yield curves is particularly beneficial for emerging markets since such markets do not have sufficient (usually non-existent) long term financial instruments.

2.4.5 Yield curve extrapolation

Thomas (2008) highlights the fact that the South African term structure of interest rates maximum maturity date is 30 years. However, insurance companies and pension funds are exposed to interest rate risk beyond 30 years. This raises the following problems: the traditional interest rate hedging tools fail to work, the is no observable yield curve data points beyond 30 years, and the long term interest rates are illiquid. Therefore, Thomas (2008) proposed alternative hedging strategies together with a number of yield curve extrapolation techniques which determine the unobservable spot rates at maturities larger than the maximum observable maturity date of bonds in the South African market. Among the extrapolation techniques were the simple techniques which included the final forward rate, linear forward rate, exponential forward rate and the final spot rate technique etc. More advanced extrapolation techniques included the Nelson-Siegel, Svensson, Cairns and the Smith Wilson methods. Thomas (2008) concluded that the Smith Wilson extrapolation method provided the best results for hedging long term exposures.

Balter, Pelsser & Schotman (2013) focussed on the uncertainty of the various extrapolation techniques. The research aimed to extrapolate the term structure of interest rates up to 100 years. The extrapolation techniques compared were the Affine Vasiek (Bayesian approach), Nelson-Siegel, Smith Wilson and an ultimate forward rate was included. The results revealed that the Nelson-Siegel extrapolation method resulted in a flat yield curve that had a lower ultimate forward rate compared to the Smith Wilson method. Also, the Nelson-Siegel method was the most volatile extrapolation technique since changes in the quoted prices affect the entire extrapolated curve. The next section will briefly discuss zero rates forecasting.

2.4.6 Yield curve forecasting

Matsuda, Tsukuda & Ullah (2003) compared the Nelson-Siegel and the Cox-Ingersoll-Ross (CIR) models in terms of their ability to fit the Japanese in sample yield curve data and if that then implies a good out of sample forecast. The findings were that the Nelson-Siegel model performed better than the CIR for both fitting and forecasting the yield curve. Moreover, the simulated Nelson-Siegel model also produced stylised attributes of a typical yield curve; the CIR could not. Finally, the Nelson-Siegel non-linear model (varying λ in the estimation procedure) outperformed the Nelson-Siegel linear model (fixing λ in the estimation procedure) in both fitting and forecasting the Japanese yield curve.

2.5 SUMMARY

To summarise, the Nelson & Siegel (1987) parsimonious model for the estimation of a yield curve and some of its historical adaptations and applications were reviewed. Furthermore, the model was improved to make the estimated latent factors dynamic; also to allow for the incorporation of macroeconomic variables. This improved version of the model can reproduce the stylised shapes of the yield curve which are backed up by economic theory: Market segmentation, Pure expectations and the Liquidity preference theory. Now, this model will be applied to the South African zero yield curve.

CHAPTER 3
30 YEAR ZERO YIELD CURVE MODELLING AND FORECASTING

3.1 INTRODUCTION

"Students of statistical demand functions might find it more productive to examine how the whole term structure of yields can be described more compactly by a few parameters." (Friedman ,1977)

The proclamation above made by Nobel prize winner Milton Friedman highlights the essential need to parameterise the term structure. This too was noted by Nelson & Siegel (1987) and so this chapter will attempt to parameterise the South African zero yield curve. Moreover, this chapter will examine the behaviour of the estimated parameters obtained from the zero yield curve over time by using the Nelson & Siegel (1987) model. Subsequently, an attempt will be made to examine how these estimated parameters interact with certain macroeconomic variables and if these parameters will add value to the forecasting of the zero yield curve.

3.2 DATA DESCRIPTION AND METHODOLOGY

3.2.1 Data Description

This chapter provides a method to apply the Nelson & Siegel (1987) model to the South African fixed income market. Methods described by Diebold & Li (2006) will be used in order to characterise the behaviour of the zero yield curve over time. Therefore, historical zero rates data are required. For the purposes of this paper, month end BEASA zero rates of maturities 3, 6,12,15,18, 24, 30, 36, 48, 60, 72, 84, 96, 108, 120, 144,180, 240, 300 and 360 months were obtained from the Johannesburg Stock exchange for the period ranging from February 2004 to June 2016. These rates are plotted in Figure 3.1. This curve, over time, appears to be upward sloping on average. However, during the period between 2007 to 2009, the zero yield curve seems to be downward sloping. This is expected because inverted yield curves normally occur during periods of financial stress. Another observable attribute of this plot is that zero rates at long term maturities are less variable than at short term maturities.

Figure 3.1: 30 year BEASSA yield curve

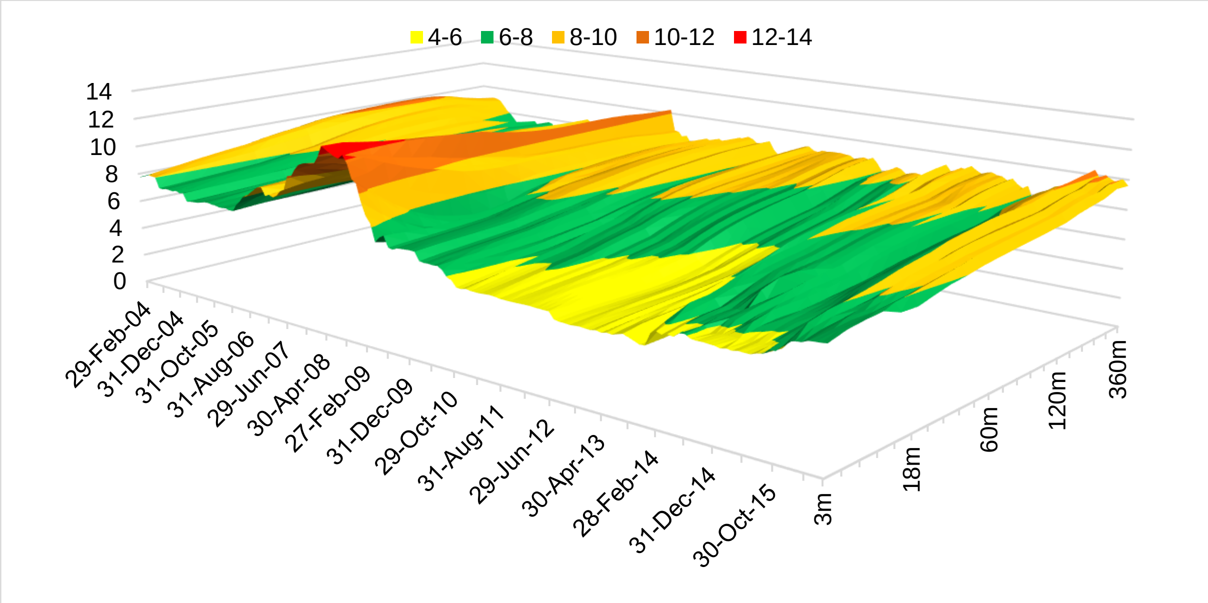


Table 3.1: Summary statistics of zero rates

Maturity (Months)	Mean	Std.dev	Min	Max	$\hat{\rho}(1)$	$\hat{\rho}(12)$	$\hat{\rho}(30)$
3	7.13	1.96	4.80	12.63	0.989	0.611	-0.01
6	7.15	1.84	4.70	12.60	0.985	0.595	0.007
9	7.17	1.70	4.67	12.37	0.98	0.572	0.02
12	7.18	1.59	4.78	12.16	0.975	0.55	0.037
15	7.21	1.51	4.97	12.01	0.969	0.514	0.041
18	7.26	1.45	4.95	11.92	0.962	0.478	0.038
24	7.36	1.35	4.98	11.83	0.953	0.435	0.038
30	7.46	1.26	5.09	11.71	0.946	0.41	0.037
36	7.56	1.17	5.24	11.57	0.94	0.39	0.032
48	7.74	1.04	5.50	11.28	0.928	0.345	0.003
60	7.89	0.95	5.68	11.04	0.917	0.286	-0.026
72	8.00	0.89	5.82	10.86	0.907	0.215	-0.052
84	8.10	0.84	5.93	10.75	0.9	0.168	-0.089
96	8.19	0.81	6.05	10.70	0.892	0.133	-0.125
108	8.25	0.78	6.18	10.69	0.886	0.106	-0.155
120	8.29	0.75	6.31	10.69	0.882	0.083	-0.183
144	8.37	0.70	6.60	10.66	0.873	0.038	-0.245
180	8.45	0.66	6.99	10.57	0.861	0.014	-0.286
240	8.45	0.65	7.02	10.43	0.864	0.085	-0.171
300	8.43	0.70	6.73	10.34	0.878	0.181	-0.034
360	8.40	0.73	6.51	10.33	0.887	0.253	0.034

3.2.2 Methodology

The case where the decay factor, λ, is kept fixed over the time period range will be considered first. The Diebold & Li (2006) DNS framework with the two step estimation procedure will be applied. To obtain the value of λ that will remain constant over the sample period, (3.1), which is the curvature loading factor (L_c) at some medium term maturity, will be maximised. Note that a constant λ implies constant loading factors over the whole sample period.

$$L_c = \left[\frac{1 - e^{-\lambda\tau}}{\lambda\tau - e^{-\lambda\tau}}\right] \tag{3.1}$$

For various medium term maturities, different values of λ are obtained by maximising (3.1) and then the sum of squared residuals ($\mathcal{R}^2$) is calculated as follows :

$$\mathcal{R}^2 = \sum_{t=1}^{T} [y_t(\tau) - \hat{y}_t(\tau)]^2 \tag{3.2}$$

Where

- $y_t(\tau)$ is the observed zero rate at maturity τ at time t
- $\hat{y}_t(\tau)$ is the estimated zero rate at maturity τ at time t
- T is the number observations

Afterwards, the λ that minimises $\mathcal{R}^2$ over the whole sample period will be chosen. This value of λ will then be used to estimate the latent factors. Now due to the fact that the loading factors are constant over time, the method of ordinary least squares regression will be used to estimate the latent factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ for each of the sample zero yield curves. This is the first step of the estimation procedure. Therefore $\widehat{l_t}, \widehat{s_t}$ and $\widehat{c_t}$ form 3 sets of time series data and are associated with the empirical approximations defined by Diebold & Li (2006) as:

$$\mathbb{P}_l = y(\infty) \tag{3.3}$$

$$\mathbb{P}_s = y(0) - y(\infty) \tag{3.4}$$

$$\mathbb{P}_c = [y(\psi) - y(\infty)] - [y(0) - y(\psi)] \tag{3.5}$$

Where

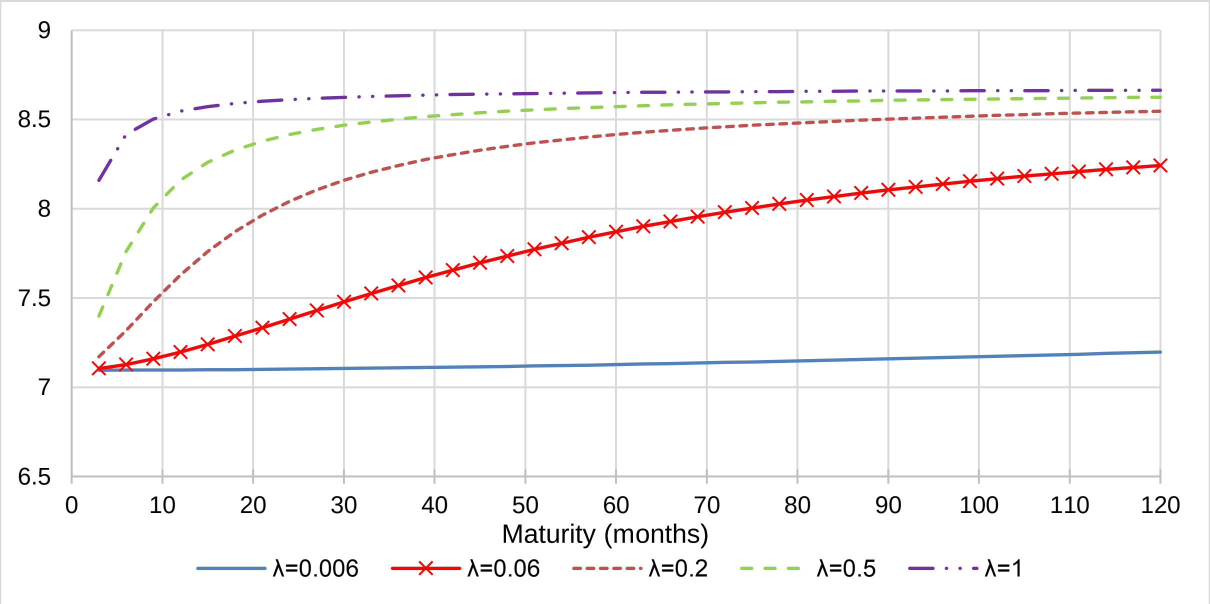
- $\{\mathbb{P}_l, \mathbb{P}_s, \mathbb{P}_c\}$ are the respective approximations for factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$.
- $\{\, y(\infty), y(\psi), y(0)\}$ are the respective observed short, medium and long term zero rates.

Since the data used in this paper contains zero rates which range from 3 to 360 months' maturities, the 360 month and the 3 month zero rates will be proxies for the observed long term zero rate and the observed short term zero rate respectively. As for the medium term zero rate, its proxy will be defined as the maturity date at which the curvature loading factor is maximised. Therefore, by definition, the estimated latent factors are expected to be similar and correlated to the approximations (3.3), (3.4) and (3.5).

The second step of the two step procedure involves fitting either a first order autoregressive function AR(1) or a first order vector autoregressive function VAR(1) to the extracted time series data of the latent factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$. This will enable the latent factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ to be forecasted for out of sample time periods. Aruoba *et al.* (2006) incorporated macroeconomic variables to the VAR(1) model in order to determine if they can be used to increase the forecast accuracy of the zero yield curve. Accordingly, the South African equivalent macroeconomic variables (observed over the same time frequency as the estimated latent factors) that will be utilised together with the latent factors are**:** inflation CPI year on year change (inf_t), capacity utilization year on year change (cu_t) and the repo rate (rr_t). The data was obtained from the Bloomberg terminal. Thus, a VAR(1) model will be fitted to the following time series data: $\{l_t, c_t, s_t, inf_t, rr_t, cu_t\}$. The resulting parameter matrix of the fitted VAR(1) model will give insight regarding how the estimated latent factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ interact with the macroeconomic variables and vice versa. These interactions could lead to better out of sample zero yield curve forecasts.

For comprehensiveness, the case where λ is not fixed is also examined. Figure 3.2 is an illustration of five zero yield curves with the same estimated latent factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ but different values of λ. It is clearly evident that the decay factor λ has a large influence on the shape of the estimated zero yield curve and consequently, the assumption of a fixed λ can limit the in sample fit. Determining a distinct value of λ for each zero yield curve implies that the loading factors will varying over time. Therefore, it is no longer possible to use ordinary least square estimation but rather some non-linear numerical optimisation method will be required. For purposes of this paper, the Generalised Reduced Gradient (GRG) nonlinear method will be applied with a 10^{-5} convergence level. The case of varying λ will only be used for one purpose: to compare the in sample fit of the estimated zero yield curve for varying λ with fixed λ over the whole sample period. The root mean squared error (RMSE) over time will be used for this purpose.

Figure 3.2: Yield curves for altered decay factors λ



Finally, to analyse how the estimated latent factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ can forecast the out of sample zero yield curve 12 months in advance, the entire sample period is split up into three reduced sample periods:

- Sample period 2004:02 - 2007:12 used to forecast 2008:01 – 2008:12.
- Sample period 2008:01 – 2011:12 used to forecast 2012:01 - 2012:12.
- Sample period 2011:06-2015:06 used to forecast 2015:07 – 2016:06.

For each of these sample periods, the estimation procedure is as follows: firstly, determine the value of λ that will be kept fixed over the sample period. Secondly, use this value of λ to estimate the latent factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ by utilising the method of ordinary least square.

Afterwards, the following forecasting methods will be used:

No change $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$

Under this method, it is assumed that the estimated parameters in the last month of the sample period will remain constant over the following 12 months:

$$\begin{bmatrix} \widehat{L_t} \\ \hat{S}_t \\ \hat{C}_t \end{bmatrix} = \begin{bmatrix} \hat{L}_{t+1} \\ \hat{S}_{t+1} \\ \hat{C}_{t+1} \end{bmatrix} = \cdots = \begin{bmatrix} \hat{L}_{t+12} \\ \hat{S}_{t+12} \\ \hat{C}_{t+12} \end{bmatrix} \tag{4.1}$$

Where t is the last month in the sample.

No change (zero rates)

Under this method, it is assumed that the last observed zero rates at all maturities will remain constant over the following 12 months.

$$\begin{bmatrix} y(3)_t \\ \vdots \\ y(360)_t \end{bmatrix} = \begin{bmatrix} y(3)_{t+1} \\ \vdots \\ y(360)_{t+1} \end{bmatrix} = \cdots = \begin{bmatrix} y(3)_{t+12} \\ \vdots \\ y(360)_{t+12} \end{bmatrix} \quad (4.2)$$

Where t is the last month in the sample.

Average $\{\hat{l}_t, \hat{s}_t, \hat{c}_t\}$

Under this method, the average of the estimated latent factors $\{\hat{l}_t, \hat{s}_t, \hat{c}_t\}$ is computed over the sample period and then the assumption is that these factors will be equal to the average over the next 12 month period.

Random walk on $\{\hat{l}_t, \hat{s}_t, \hat{c}_t\}$

Under this method, random values are generated from a normal distribution with mean equal to average of the estimated latent factors and standard deviation equal to the standard deviation of the estimated latent factors. Then these randomly generated values are used as factors for the next 12 months.

$$\hat{L_t} \sim N\{mean(L_{t-1:1}), Std.dev(L_{t-1:1})\}$$
$$\hat{S_t} \sim N\{mean(S_{t-1:1}), Std.dev(S_{t-1:1})\} \quad (4.3)$$
$$\hat{C_t} \sim N\{mean(C_{t-1:1}), Std.dev(C_{t-1:1})\}$$

AR(1) on zero rates

Under this method, a first order autoregressive process AR(1) is fitted to the zero rates at all maturities. The AR(1) model is then used to forecast zero rates 12 months ahead.

$$y_t(3) = \hat{c} + \hat{\phi} y_{t-1}(3)$$
$$\vdots \quad (4.4)$$
$$y_t(360) = \hat{c} + \hat{\phi} y_{t-1}(360)$$

AR(1) on $\{\hat{l}_t, \hat{s}_t, \hat{c}_t\}$

Under this method, a first order autoregressive process AR(1) is fitted to estimated factors $\{\hat{l}_t, \hat{s}_t, \hat{c}_t\}$, over the sample period. The AR(1) model is then used to forecast factors 12 months in advance.

$$\hat{L_t} = \hat{c} + \hat{\phi} L_{t-1}$$
$$\hat{S}_t = \hat{c} + \hat{\phi} S_{t-1} \quad (4.5)$$
$$\hat{C}_t = \hat{c} + \hat{\phi} C_{t-1}$$

VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$

Under this method, a first order Vector Autoregressive process VAR(1) is fitted to the estimated factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$. The VAR(1) model is then used to forecast the factors 12 months ahead.

$$\begin{bmatrix} \widehat{L_t} \\ \widehat{S_t} \\ \widehat{C_t} \end{bmatrix} = \underline{c} + \begin{bmatrix} a_{11} & \cdots & a_{13} \\ \vdots & \ddots & \vdots \\ a_{31} & \cdots & a_{33} \end{bmatrix} \begin{bmatrix} L_{t-1} \\ S_{t-1} \\ C_{t-1} \end{bmatrix} \tag{4.6}$$

VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{inf_t}\}$

This method is similar to the method described above. The adjustment is the addition of the inflation macroeconomic variable. Therefore a VAR(1) model is fitted to the factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{inf_t}\}$. The VAR(1) model is then used to forecast factors 12 months in advance.

$$\begin{bmatrix} \widehat{L_t} \\ \widehat{S_t} \\ \widehat{C_t} \\ \widehat{inf_t} \end{bmatrix} = \underline{c} + \begin{bmatrix} a_{11} & \cdots & a_{14} \\ \vdots & \ddots & \vdots \\ a_{41} & \cdots & a_{44} \end{bmatrix} \begin{bmatrix} L_{t-1} \\ S_{t-1} \\ C_{t-1} \\ inf_{t-1} \end{bmatrix} \tag{4.7}$$

VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{inf_t}, \widehat{ccu_t}, \widehat{rr_t}\}$

This method too is similar to the method described above. The adjustment is the addition of the capacity utilisation and the repo rate macroeconomic variables. Therefore a VAR(1) model is fitted to the factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{inf_t}, \widehat{ccu_t}, \widehat{rr_t}\}$.The VAR(1) model is then used to forecast the factors 12 months ahead:

$$\begin{bmatrix} \widehat{L_t} \\ \widehat{S_t} \\ \widehat{C_t} \\ \widehat{inf_t} \\ \widehat{CU_t} \\ \widehat{RR_t} \end{bmatrix} = \underline{c} + \begin{bmatrix} a_{11} & \cdots & a_{16} \\ \vdots & \ddots & \vdots \\ a_{61} & \cdots & a_{66} \end{bmatrix} \begin{bmatrix} L_{t-1} \\ S_{t-1} \\ C_{t-1} \\ inf_{t-1} \\ CU_{t-1} \\ RR_{t-1} \end{bmatrix} \tag{4.8}$$

In order to assess how well forecasting methods described above performed, the RMSE as calculated as in (4.9) will be used. This measure will be used asses the mean error between the actual observed zero rates and those forecasted.

$$RMSE = \sqrt{\frac{1}{n} \times \sum_{i=1}^{n} [y_t(\tau) - \hat{y}_t(\tau)]^2} \tag{4.9}$$

Where

- $y_i(\tau)$ is the observed zero rate at maturity τ
- $\hat{y}_i(\tau)$ is the forecasted zero rate at maturity τ.
- n is the number of maturities considered
- t is the forecasted period

These different forecasting methods are then grouped into four categories and then the categories are compared to one another in order to evaluate whether adding macroeconomic variables to the estimated latent factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ will improve the forecasts.

The various categories are defined as follows:

Category 1: No change (zero rates), Random walk on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$, AR(1) on zero rates.

Category 2: Average $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$, No change $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$, AR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$

Category 3: VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ (with and without exogenous shock).

Category 4: VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{inf_t}\}$, VAR(1) on $\{\widehat{\mathrm{l}_t}, \widehat{\mathrm{s}_t}, \widehat{\mathrm{c}_t}, \widehat{\mathrm{inf}_t}, \widehat{\mathrm{ccu}_t}, \widehat{\mathrm{rr}_t}\}$ (with and without exogenous shock).

3.3 RESULTS

3.3.1 Estimation with fixed λ

As discussed in the previous section, a value for λ first has to be determine such that the sum of squared residuals ($\mathcal{R}^2$) is minimised over the whole sample period. Table 3.2 provides the values of λ that were obtained through maximising the curvature loading factor at different medium term maturities. Each medium term maturity date corresponds to a unique value of λ. It is established that $\lambda = 0.0609$, as proposed by Diebold Li (2006), results in the lowest $\mathcal{R}^2$ and thus this value of λ will be kept fixed over the entire sample period. A plot of the loading factors for $\lambda = 0.0609$ is provided in Figure 3.3.

Table 3.2: λ estimation results

max curve loading at maturity	λ	$\mathcal{R}^2$
18	0.0996	114.12
24	0.0747	99.10
28	0.0640	96.82
29	0.0618	96.72
30	0.0597	96.74
31	0.0578	96.87
33	0.0543	97.35
36	0.0498	98.44
40	0.0448	100.20
48	0.0373	103.72
60	0.0298	107.95
72	0.0249	111.27
84	0.0213	114.28
Diebold li	0.0609	96.71

Figure 3.3: loading factors for λ=0.0609

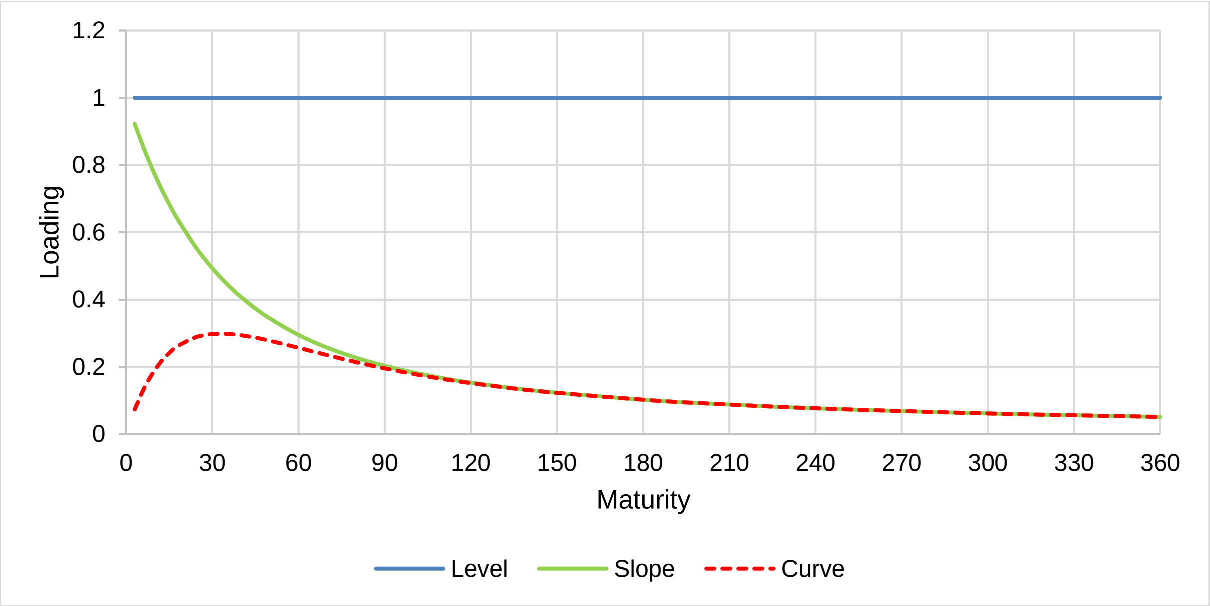
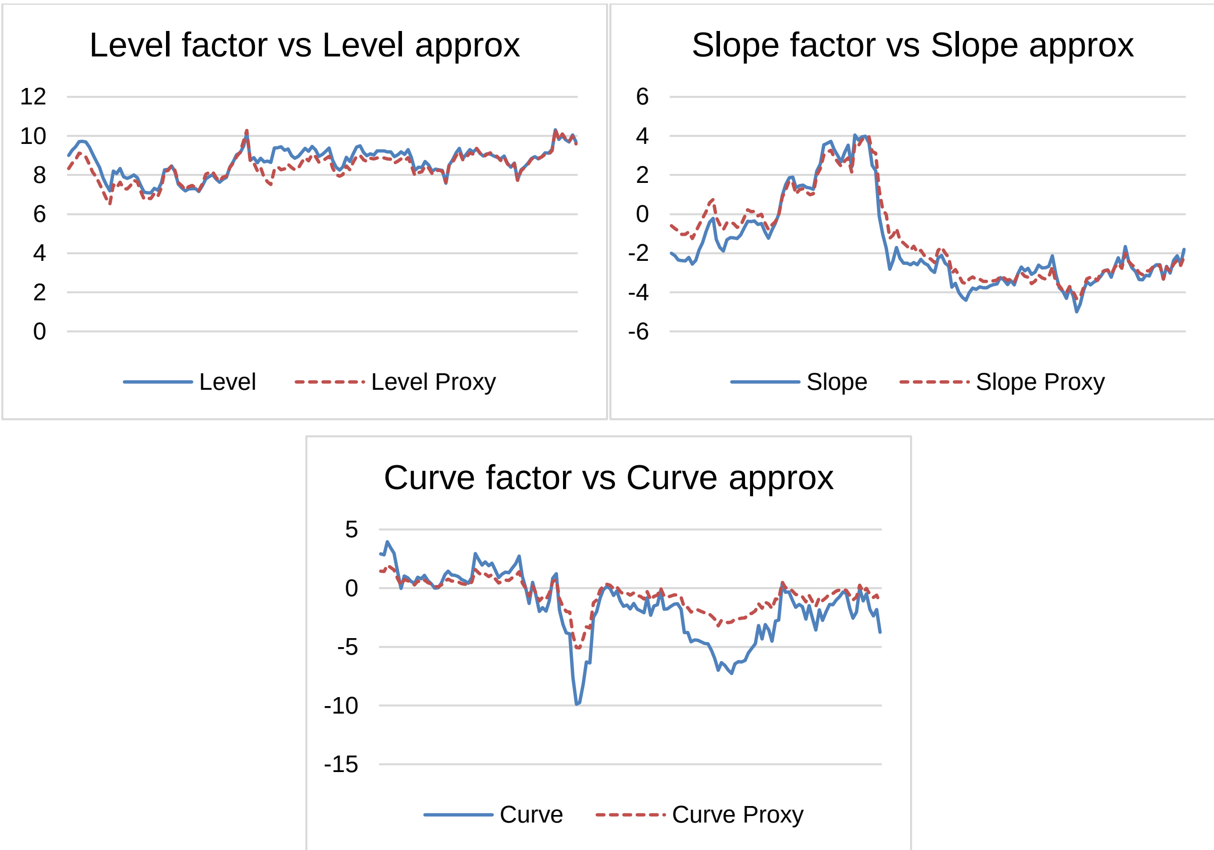


Table 3.3: Summary statistics estimated factors

Factor	Mean	Std.dev	Minimum	Maximum	$\hat{\rho}(1)$	$\hat{\rho}(12)$	$\hat{\rho}(30)$	ADF (t-statistics)	ADF (p-value)
Level	8.67	0.73	7.08	10.31	0.892	0.266	-0.093	-2.63	0.0882
Slope	-1.58	2.27	-5.00	4.04	0.976	0.619	0.019	-1.10	0.2426
Curve	-1.55	2.78	-9.88	3.95	0.924	0.349	0.038	-1.87	0.0581

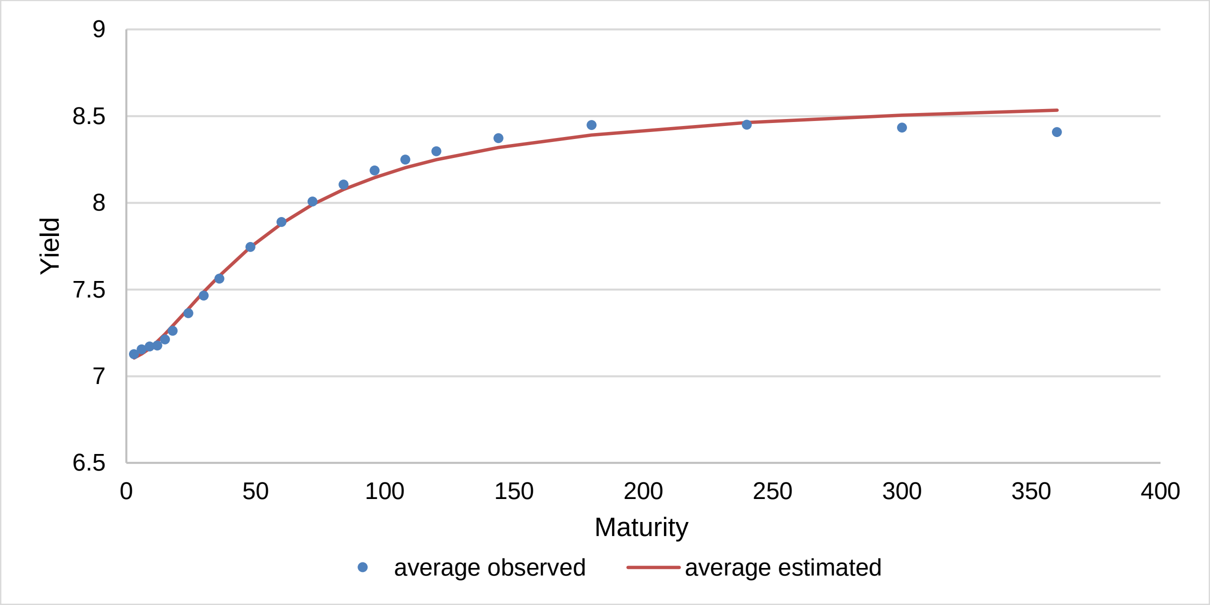
Figure 3.4: Estimated factors vs. approximated factors



As discussed in the previous section, Diebold & Li (2006) suggested some empirical approximations that the estimated latent factors can be associated with and these approximations are provided in (3.2), (3.3) and (3.4). In Figure 3.4, the approximations $\{\mathbb{P}_l, \mathbb{P}_s, \mathbb{P}_c\}$ are plotted against the latent estimated factors $\{\hat{l}_t, \hat{s}_t, \hat{c}_t\}$. By observation, it is clear that the estimated factors closely match their empirical approximations. The correlations between estimations and approximations are: $corr(\hat{l}_t, \mathbb{P}_l) = 0.908, corr(\hat{s}_t, \mathbb{P}_s) = 0.969$ and $corr(\hat{c}_t, \mathbb{P}_c) = 0.992$. The calculated correlations show a high degree of comparison between the estimated factors and approximated factors. Diebold & Rudebusch (2013:9) mentions that this implies that different factors might be related to specific macroeconomic influences.

The estimated latent factors are then used to estimate a zero yield curve. For a goodness of fit test, in Figure 3.5, the average values of the zero yields are plotted against the curve of the average latent estimated factors. It is visually clear that on average, the Nelson & Siegel (1987) model does provide a good fit over the period of February 2004 through June 2016. Although the estimated curve does not pass through all of the observed yields, it accurately reproduces the overall shape of the zero yield curve.

Figure 3.5: Average yield curve 2004:02-2016:06



Another way to evaluate the goodness of fit is to plot the residuals over time for all maturities. This is achieved by taking the difference between the observed and the estimated zero rates. This is then plotted in Figure 3.6. Most residuals fall in the interval of $[-0.2, 0.2]$ but for certain periods (such as 2004/05, 2008/09 and 2014) some curves have larger residuals in the interval of $[-1, 1]$. To determine why larger errors are obtained during these periods, sample months are taken for these periods and then plotted against their observed and estimated zero rates. This is shown in Figure 3.7. For the zero curve of 30 September 2004, the curvature maximises at maturity of 100 months. This means that the estimation error is due to keeping λ fixed. For the zero yield curve of 30 January 2009, the error in estimation is due to the observed zero rates having two humps whereas the Nelson & Siegel (1987) model only makes provision for a single humped zero yield curve. Finally, for the case of 31 July 2014, there is a spike in zero rates around the 24 month maturity that, again, is not incorporated by the Nelson & Siegel (1987) model. The above cases are evidence that for certain circumstances, like for example assuming a fixed λ and when multiple humps are observed, the Nelson & Siegel (1987) model might lead to an inappropriate fit. However, because such cases occur irregularly (are exceptions), it can be concluded that the estimated zero yield curves appear to fit well overall.

Figure 3.6: 30 year yield curve residuals

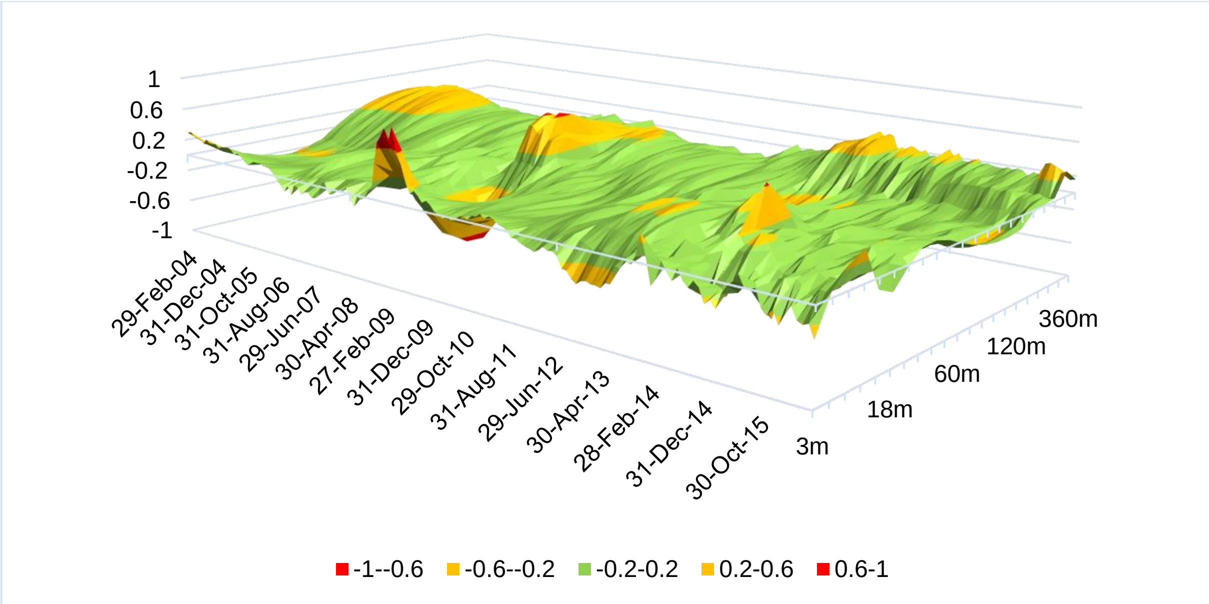
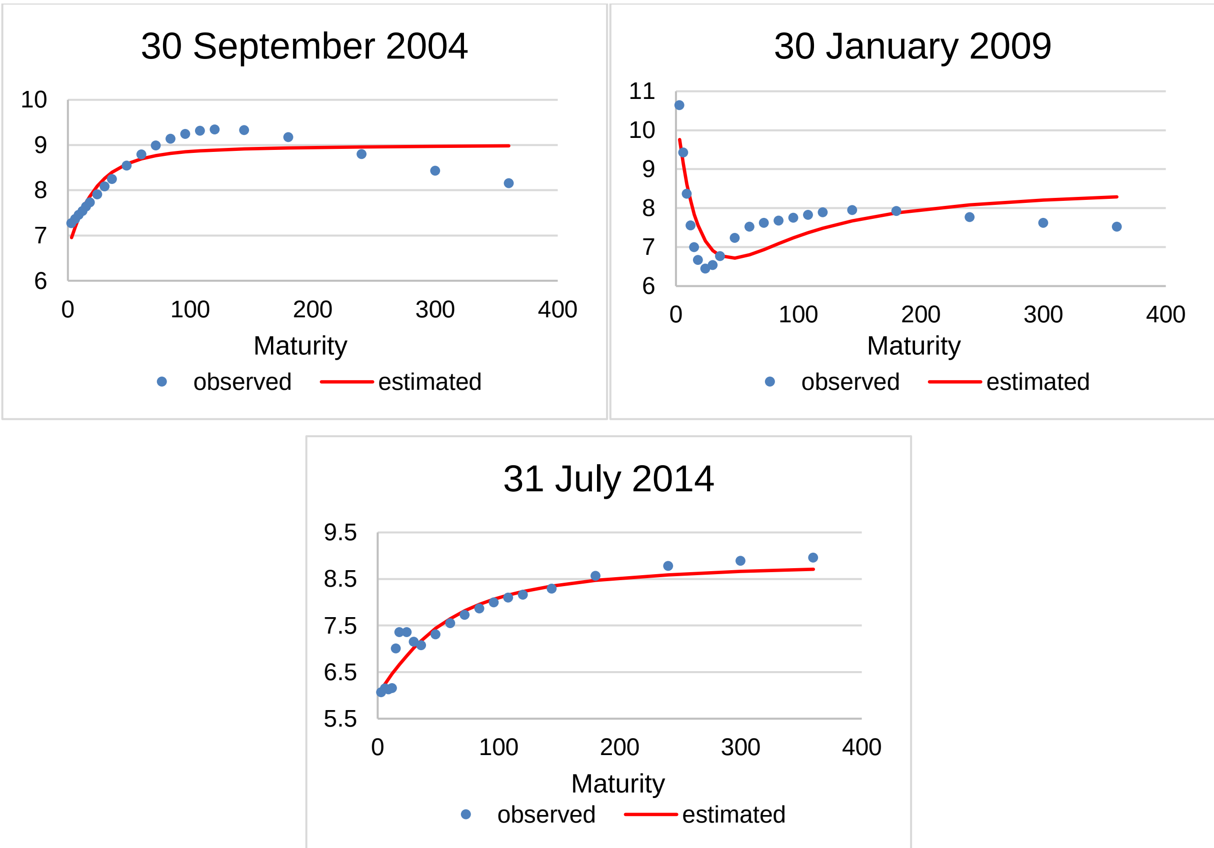


Figure 3.7: 30 year yield curve fit



3.3.2 Estimation with varying λ

As discussed in the methodology section, it might be beneficial to contrast the assumption of maintaining a fixed λ throughout the sample with varying λ. Therefore, all the latent factors together with λ_t, $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{\lambda_t}\}$, are estimated and then a goodness of fit needs to be evaluated. Again, in Figure 3.8 the residuals are plotted over time for all maturities. By just assessing Figure 3.8 visually, compared to Figure 3.6, it is apparent that varying λ over time brings about a superior fit. In Figure 3.8, most residuals fall in the interval $[-0.2, 0.2]$. Again, in the periods of 2004/05, 2008/09 and 2014, there are slightly larger residuals than the average. However, compared to case where λ is constant, these residuals are substantially lower. Once more, similar to Figure 3.7, Figure 3.9 contains the plot of the sample observed and estimated zero yield curves for varied λ. These estimated zero yield curves generally fit better for the varying λ case with the only exception being the 31 July 2014 zero yield curve which has an additional hump, that is not incorporated by the Nelson & Siegel (1987) model, around the 24 month maturity.

Figure 3.8: 30 yield curve residuals with varying λ

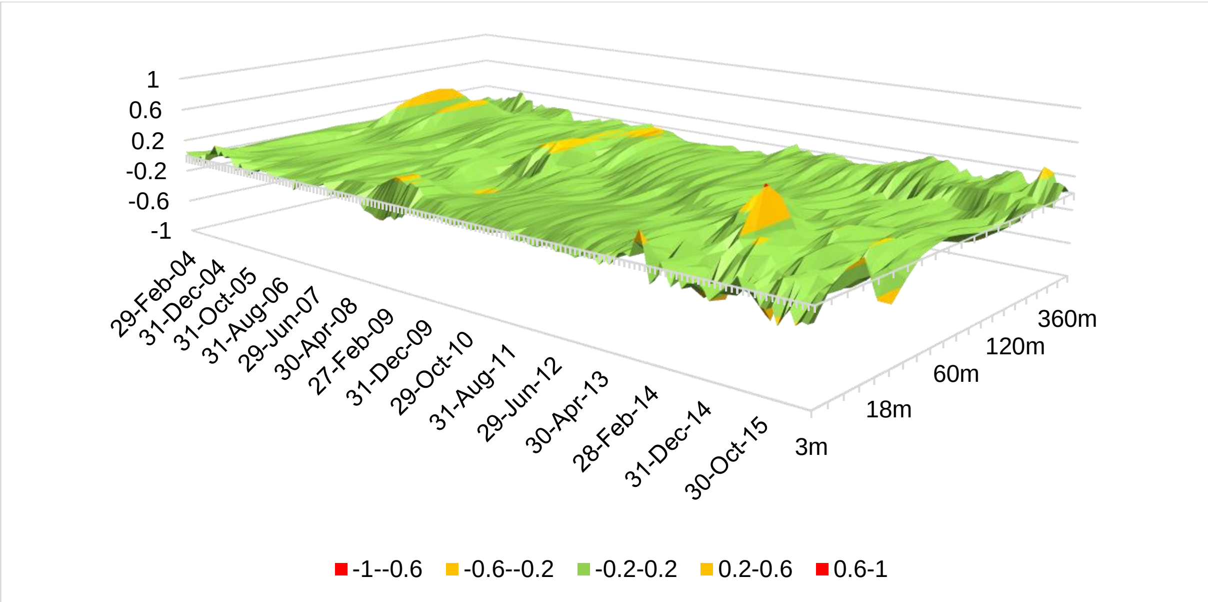
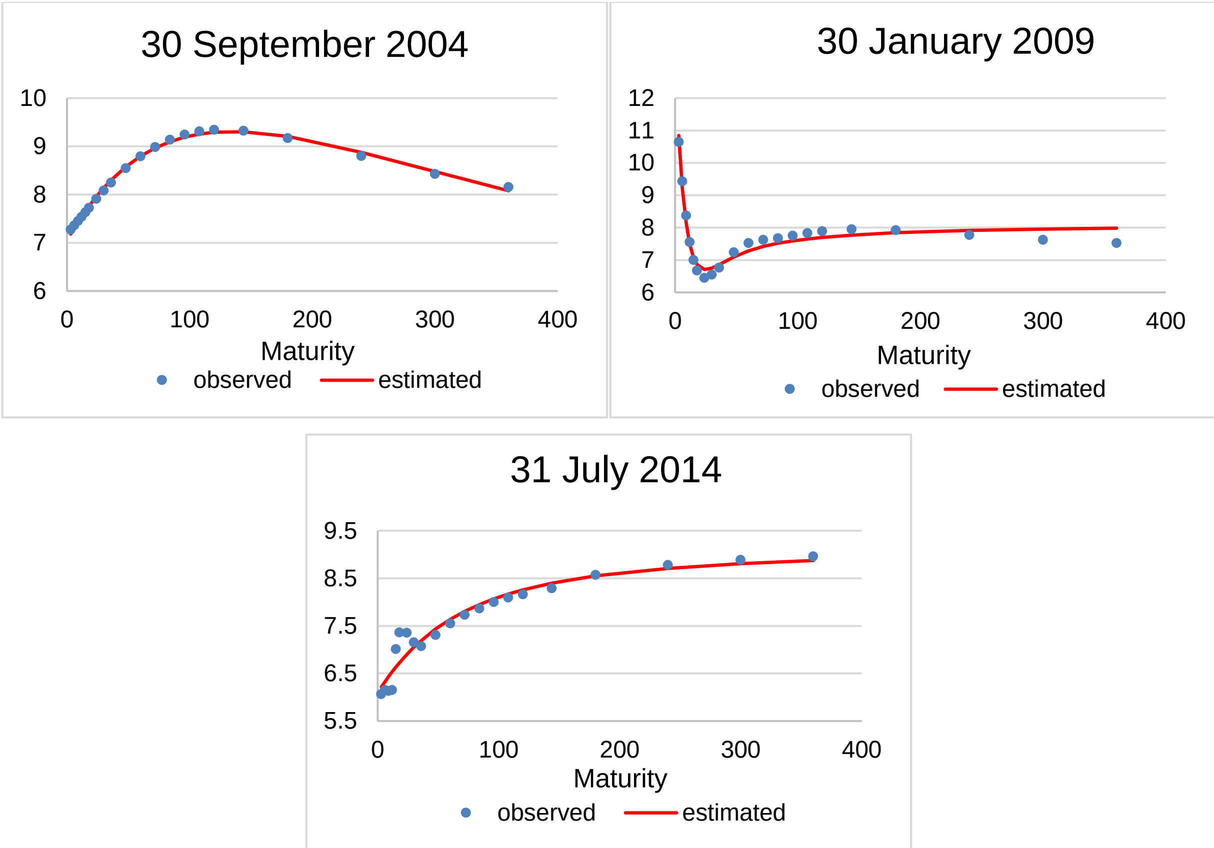
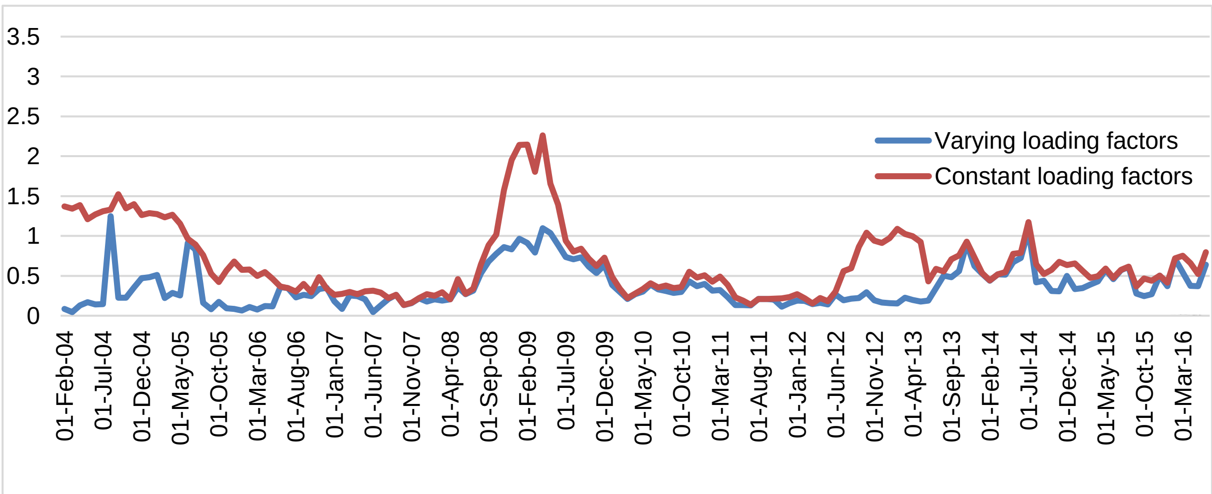


Figure 3.9: 30 yield curve plots



Finally, to compare the fit of the model for the case when λ is constant to the case when λ is varying, the squared residuals are plotted for both cases over time in Figure 3.10. It is clear that the case of varying λ provides squared residuals that are at the worst, as high as when λ is constant. It is also apparent that squared residuals for both cases are somewhat correlated. This could imply that there are external factors, other than λ, that cause estimation error.

Figure 3.10: Squared residuals over time



Hence, it is clear that varying λ, as opposed to keeping it fixed, provides a better in sample fit. This average better fit is especially apparent for yield curves that have irregular shapes and occur infrequently. Varying λ has it's disadvantages: loading factors are not constant over the sample period, and consequently, the estimated latent factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ do not have relatable values over time. To forecast out of sample, relatable values over time are required. Additionally, the latent estimated factors do not seem to have a strong relationship with the approximations described by equations (3.3), (3.4) and (3.5). Another possible obstacle to varying λ, is that ordinary least squares cannot be applied to estimate the latent factors. Instead, numerical optimisation methods, which are more computationally intensive, are required.

3.3.3 Relationship with macro variables

For this section, the factors estimated in section 3.2.1, with the assumption of constant λ, will be used. As discussed in the methodology, the time series of factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ are combined with observed macroeconomic variables over the same period. Moreover, a VAR(1) process is fitted on these set of time series data of variables. The estimated parameter matrix of the VAR(1) model will then be analysed in order to determine whether macro-latent factor and lagged values interactions exist.

Table 3.4 provides the resultant estimated parameter matrix for the VAR(1) process. All values displayed in bold text are significant at 1% significance level by means of the t-test; the numbers in brackets are the standard errors of these values. Note that all diagonal values are significant. This is expected because all these factors are dependent on their lagged values. Also, observe that slope factor is influenced by the lagged values of curve factor; the slope factor also has a significance with inflation and capacity utilisation. This provides some indication of a macro-to-yield relationship. On the other hand, analyses of the interaction between capacity utilisation and the repo rate with all estimated latent factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ reveal evidence of a yield-to-macro relationship. Despite the existence of a bidirectional relationship, the yield-to-macro relationship is stronger than the macro-to-yield. The information provided by these relationships may enhance the out of sample forecasts. It is important to note that for the purposes of this paper a 1% significance level was used to assess the significance of the interaction; this is dissimilar to Aruoba *et al.* (2006) which used a 5% significance level. To find interactions that are more robust and meaningful, a more restrictive approach is applied in this paper. The significance of these interaction will again be tested when this VAR(1) model is used to forecast the estimated latent factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$, for out of sample observations.

Table 3.4: VAR(1) parameter estimation results

	LEVEL	SLOPE	CURVE	INF	CU	RR
LEVEL(-1)	**1.135289** (0.08825)	-0.040356 (0.11419)	-0.436005 (0.25839)	0.228756 (0.11391)	**0.427412** (0.12365)	**0.298265** (0.05710)
SLOPE(-1)	0.133275 (0.08175)	**0.926300** (0.10577)	-0.433156 (0.23935)	0.211822 (0.10552)	**0.321539** (0.11454)	**0.271606** (0.05289)
CURVE(-1)	-0.013068 (0.01274)	**0.102773** (0.01648)	**0.895531** (0.03729)	0.029165 (0.01644)	**0.050601** (0.01784)	**0.059189** (0.00824)
INF(-1)	-0.063142 (0.03003)	**0.158632** (0.03885)	-0.080253 (0.08791)	**0.954320** (0.03875)	-0.145062 (0.04207)	0.029236 (0.01943)
CU(-1)	-0.036066 (0.01559)	**0.063765** (0.02017)	-0.049146 (0.04563)	0.011244 (0.02012)	**0.892202** (0.02184)	0.013306 (0.01008)
RR(-1)	-0.085114 (0.07444)	-0.073443 (0.09631)	0.471547 (0.21794)	-0.18534 (0.09608)	-0.317465 (0.10429)	**0.685719** (0.04816)

3.3.4 Forecasting results

As formerly described in section 3.1, once all the latent factors have been extracted and their interactions with the relevant macroeconomic variables has been evaluated, this information can be used to make forecasts of the zero yield curve. For purposes of this paper, the forecasting period is reduced to three distinct periods and the forecasting performance is judged based on the 1, 6 and 12 month ahead RMSE.

Table 3.5: forecasting for 2008:01 – 2008:12

Method	**2008 (RMSE)**		
Forecast period:	**1 month**	**6 month**	**12 month**
No change (zero rates),	0.222	2.132	1.676
Random walk on $\{\hat{l}_t, \hat{s}_t, \hat{c}_t\}$	2.733	3.776	2.903
AR(1) on zero rates	0.215	2.261	1.354
Category 1 average RSME	1.474	2.723	1.977
Average $\{\hat{l}_t, \hat{s}_t, \hat{c}_t\}$	1.436	3.185	1.057
No change $\{\hat{l}_t, \hat{s}_t, \hat{c}_t\}$	0.328	2.126	1.753
AR(1) on $\{\hat{l}_t, \hat{s}_t, \hat{c}_t\}$	0.374	2.709	1.078
Category 2 average RSME	0.713	2.673	1.296
VAR(1) on $\{\hat{l}_t, \hat{s}_t, \hat{c}_t\}$	0.343	1.376	3.832
VAR(1) on $\{\hat{l}_t, \hat{s}_t, \hat{c}_t\}$ +C	0.370	1.890	3.292
Category 3 average RSME	0.357	1.633	3.562
VAR(1) on $\{\hat{l}_t, \hat{s}_t, \hat{c}_t, \widehat{inf}_t\}$	0.342	1.380	3.746
VAR(1) on $\{\hat{l}_t, \hat{s}_t, \hat{c}_t, \widehat{inf}_t\}$ +C	0.377	1.885	3.266
VAR(1) on $\{\hat{l}_t, \hat{s}_t, \hat{c}_t, \widehat{inf}_t, \widehat{ccu}_t, \widehat{rr}_t\}$	0.331	0.933	4.610
VAR(1) on $\{\hat{l}_t, \hat{s}_t, \hat{c}_t, \widehat{inf}_t, \widehat{ccu}_t, \widehat{rr}_t\}$ + C	0.387	2.226	2.648
Category 4 average RSME	0.359	1.606	3.568

In Table 3.5, the forecasted period of 2008 is assessed. For the 1month ahead forecast, the no change and the AR(1) on the zero rates performed the best. This is expected since the zero rates have a high correlation with their 1 month lagged value. On the other hand, for the 1 month ahead forecast, models that included macroeconomic variables underperformed models that did not include them.

For the 6 month ahead forecast, the VAR(1) on the $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{inf_t}, \widehat{ccu_t}, \widehat{rr_t}\}$ model performed the best, followed by the VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$, while the worst performance was the random walk. The mean value of the RMSE for each category reveal that category 4 models performed the best, category 3 came second while the worst performing category was 1. Note though, that there is only a marginal difference between categories 3 and 4 models' performance.

For the 12 month ahead forecast, the Average of $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ model performed the best; then the VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ ranked second while the worst performing model was the VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{inf_t}, \widehat{ccu_t}, \widehat{rr_t}\}$. The mean value of the RMSE for each category reveal that category 2 models performed the best, category 1 came second while the worst performing category was 4. Note again that there is only a marginal difference between categories 3 and 4 models' performance.

Thus in reviewing the forecasting results for 2008, it is clear that the 1 month ahead forecast, assuming no change for the term structure of interest rates, provides better forecasts than any of the models which use the estimated latent factors or macroeconomic variables. However, when the 6 and 12 month forecasts are assessed, it can be concluded that using the latent factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ and macroeconomic variables, provide meaningful forecasts. Consequently, it is important to note that including macroeconomic variables does significantly increase the forecast accuracy for the period of 2008.

In Table 3.6, the forecasted period of 2012 is assessed. Once again, it is clear that the 1 month ahead forecast when assuming the no change or when using an AR(1) on the zero rates performs best. As for the 6 month ahead forecast, the VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{inf_t}, \widehat{ccu_t}, \widehat{rr_t}\}$ + *c* model performed the best. It is interesting to note that over the 6 month forecasted period, the inclusion of macroeconomic variables does not produce better forecasts.

Table 3.6: Forecast for 2012:01 – 2012:12

Method	2012 (RMSE)		
Forecast period:	**1 month**	**6 month**	**12 month**
No change (zero rates)	0.177	0.584	1.023
Random walk on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$	1.214	0.594	3.782
AR(1) on zero rates	0.250	0.938	1.543
Category 1 average RSME	0.547	0.705	2.116
Average $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$	1.397	1.783	2.150
No change $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$	0.256	0.623	1.062
AR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$	0.311	1.031	1.734
Category 2 average RSME	0.655	1.146	1.649
VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$	0.274	0.688	1.081
VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ + C	0.283	0.751	1.300
Category 3 average RSME	0.279	0.720	1.191
VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{inf_t}\}$	0.275	0.809	1.185
VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{inf_t}\}$ + C	0.360	1.098	1.679
VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{inf_t}, \widehat{ccu_t}, \widehat{rr_t}\}$	0.241	0.591	1.183
VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{inf_t}, \widehat{ccu_t}, \widehat{rr_t}\}$ + C	0.277	0.576	1.180
Category 4 average RSME	0.288	0.769	1.307

On the other hand, for the 12 month ahead forecast, the No change (zero rates) model performed the best; the No change $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ ranked second, and the worst performing was the random walk. The mean value of the RMSE for each category reveal that category 3 models performed the best, category 4 models came second and the worst performing category was 1. Once again, a marginal difference is observed between models that use macroeconomic variable.

Table 3.7: Forecast for 2015:07 – 2016:06

Method	2015/2016 (RMSE)		
Forecast period:	**1 month**	**6 month**	**12 month**
No change (zero rates),	0.111	1.518	0.888
Random walk on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$	0.423	2.508	1.993
AR(1) on zero rates	0.733	2.164	1.269
Category 1 average RSME	0.422	2.063	1.383
Average $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$	0.737	2.165	1.468
No change $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$	0.159	1.520	0.884
AR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$	0.249	1.683	1.080
Category 2 average RSME	0.382	1.789	1.144
VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$	0.311	1.882	1.292
VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ + C	0.235	1.875	1.331
Category 3 average RSME	0.273	1.879	1.312
VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{inf_t}\}$	0.252	1.720	1.102
VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{inf_t}\}$ +C	0.267	1.845	1.288
VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{inf_t}, \widehat{ccu_t}, \widehat{rr_t}\}$	0.264	1.719	1.066
VAR(1) on $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{inf_t}, \widehat{ccu_t}, \widehat{rr_t}\}$ + C	0.286	1.987	1.448
Category 4 average RSME	0.267	1.818	1.226

The final forecasted period considered is the July 2015 to June 2016 period presented in Table 3.7. An interesting observation is made when comparing the 12 month and 6 month RMSE over a longer forecasted period of 12 months. As expected, the is a significantly lower RMSE compared to the 6 month RMSE. This might be due to only the particular period observed where the 12 month ahead zero yield curve had similar behaviour to what was observed in the sample period.

3.4 SUMMARY

The South African zero yield curve between the sample period exhibited the expected shape: upward sloping as maturity increases, inverted shape in times of financial stress and less variability for the long term maturities. Moreover, keeping λ fixed resulted in an easier estimation procedure (linear) as opposed to varying it (non-linear) for the latent factors. However, the varied λ approach resulted in a better in sample fit of the zero yield curve as shown by the lower squared residuals. Furthermore, since the estimated latent factors are dynamic over time, the DNS framework can be used for the forecasting of the yield curve together with AR(1) and VAR(1) models on the estimated latent factors. Macroeconomic variables together with the estimated latent factors were used to fit a VAR(1) model so that the interactions may be investigated. At 1% significance level, despite the existence of a bidirectional relationship, the yield-to-macro relationship was stronger than the macro-to-yield.

Finally, the total sample period was divided up into 3 periods to test the performance of the various forecasting methods. There is evidence that in some cases, the combination of the estimated latent factors with the macroeconomic variables may provide good forecasts. However, this cannot be resolved in this paper as there are many other factors which may have an influence on the zero yield curve that are not included in this model. Forecasting using these factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ may in future research be more valuable when an alternative forecasting procedure is applied. Nonetheless, using these factors can provide some meaningful forecasts when no other data is available.

CHAPTER 4
10 YEAR ZERO YIELD CURVE EXTRAPOLATION

4.1 INTRODUCTION

The zero yield curve, which is constructed from government bonds, has long been used as a proxy for the risk free rate. The risk free rate is a required input for many financial models (mostly to obtain the present value of future cash flows). However, a problem arises when future cash flows extend for terms greater than that of the zero yield curve. Consequently, methods are required to extend the zero yield curve past the longest observed zero rate. Thus, this chapter will describe and examine how the Nelson & Siegel (1987) model can be used to extrapolate the zero yield curve.

4.2 METHODOLOGY

In this section, the suitability of the Nelson & Siegel (1987) model for extrapolation purposes will be determined. To evaluate whether the Nelson & Siegel (1987) model accurately captures the behaviour of the long term zero rates, the realised observed zero rates at longer maturities are required. Therefore, the observed zero rates are cut off at a shorter maturity. Hence, for this chapter, it is assumed that zero rates at maturities of 3, 6, 9, 12, 15, 18, 24, 30, 36, 48, 60, 72, 84, 96, 108 and 120 months are observed. For these observed maturities, the model will be fitted and the latent factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ for each individual term structure in the sample will be estimated. Finally, these estimated latent factors will be used to extrapolate the zero yield curve to further maturities of: 144, 180, 240, 300 and 360 months.

It is necessary to define measurements which will be used to calculate the performance of the various extrapolation methods that will be used. Firstly, the fitted squared residuals ($\mathcal{R}_f^2$) will be calculated to evaluate the fit of the model for the observed maturities that will be considered, i.e. 3 to 120 month maturities. Secondly, the extrapolated squared residuals ($\mathcal{R}_E^2$) will be calculated in order to evaluate the fit of the model for the extrapolated maturities, i.e. 144 to 360 month maturities. These measures are calculated as follows:

$$\mathcal{R}_f^2 = [y_t(3) - \hat{y}_t(3)]^2 + \cdots + [y_t(120) - \hat{y}_t(120)]^2 \tag{4.1}$$

and

$$\mathcal{R}_E^2 = [y_t(144) - \hat{y}_t(144)]^2 + \cdots + [y_t(360) - \hat{y}_t(360)]^2 \tag{4.2}$$

Where

- $y_t(\tau)$ is the observed zero rate for a particular maturity τ
- $\hat{y}_t(\tau)$ is the estimated zero rate for a particular maturity τ

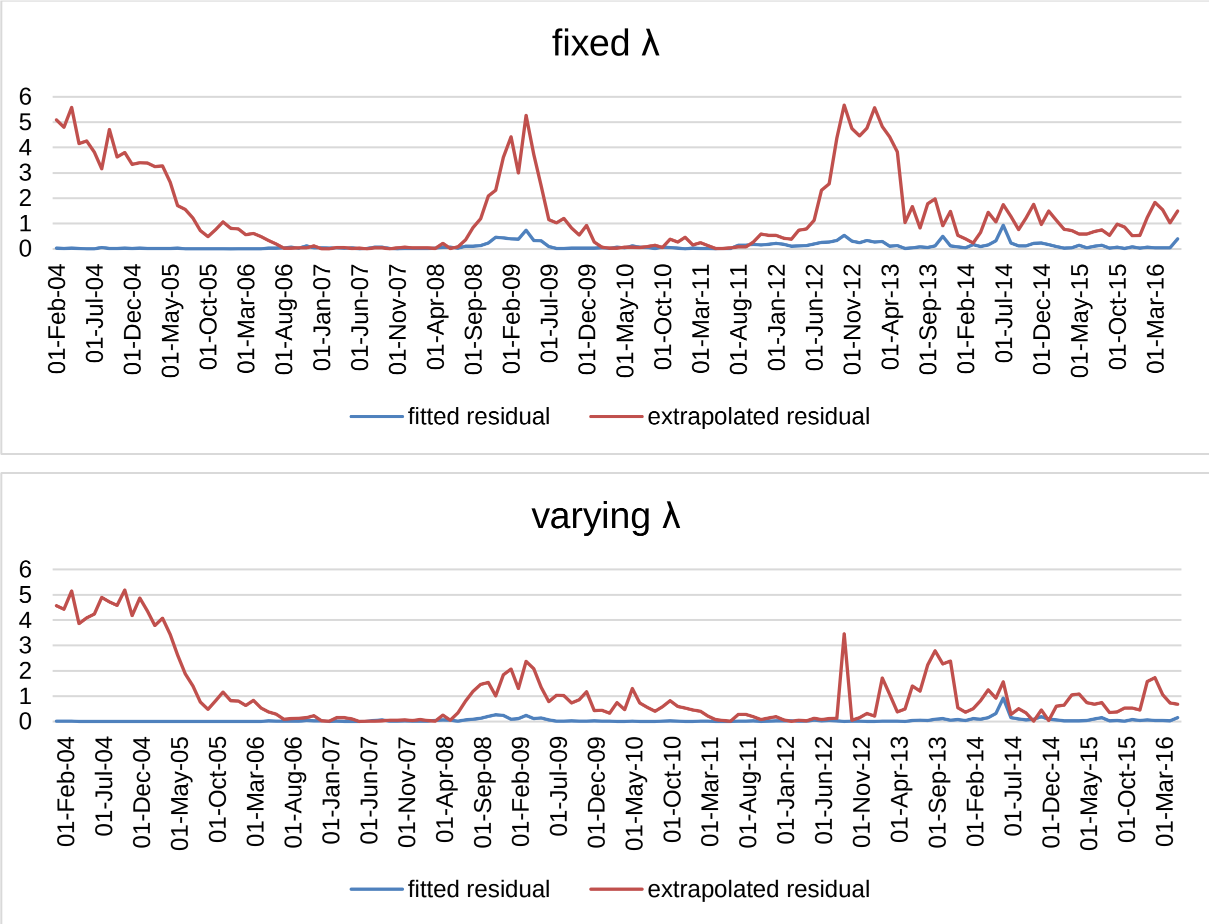
Initially, the following two methods will be considered:

- Method 1: first estimate a constant value for λ that will be applicable over the sample period February 2004 to June 2016 as described in chapter 3.1. Then estimate the latent factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ based on the 120 month zero yield curve by minimising $\mathcal{R}_f^2$.
- Method 2: again, similar to the description in chapter 3.1, all the latent factors, including the decay factor λ, are estimated for each individual curve in the sample. Hence the factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{\lambda_t}\}$ are estimated by minimising $\mathcal{R}_f^2$ for the 120 month zero yield curve for the sample period February 2004 to June 2016.

For both methods described above, $\mathcal{R}_E^2$ is calculated to evaluate the extrapolation. In Figure 4.1, $\mathcal{R}_f^2$ and $\mathcal{R}_E^2$ are plotted together over time. $\mathcal{R}_f^2$ is found to be lower over time when varying λ compared to fixing it. This outcome is analogous to chapter 3, where it was found that varying λ over time significantly improved the fit. Moreover, when $\mathcal{R}_E^2$ is compared to $\mathcal{R}_f^2$ for fixed λ, a notable positive correlation between $\mathcal{R}_f^2$ and $\mathcal{R}_E^2$ of 0.396 shows that an inaccurate in fit will lead to inaccurate extrapolation result. Counterintuitively, for the case where λ is varied, a much smaller correlation between the $\mathcal{R}_f^2$ and $\mathcal{R}_E^2$ of 0.0036 is obtained. This indicates that a good fit of the observed zero rates does not necessarily guarantee accurate extrapolated zero rates.

Therefore, because the in sample fit of the zero yield curve does not provide sufficient information that will lead to accurate extrapolations, an alternative approximation is required to estimate the long end of the yield curve. Since the long term rates are only determined by the level factor, the level factor is a possible proxy for the long term spot rates. To demonstrate, in Figure 4.2, the actual 360 month spot rate is plotted against the estimated level factor (obtained through using method 1). Despite some tracking difference/error between the level factor and the 360 month spot rate, it seems visually apparent that there is some co-movement. The correlation of 0.568 between the level factor and the 360 month spot rate suggests just that.

Figure 4.1: Fitted vs extrapolated residuals

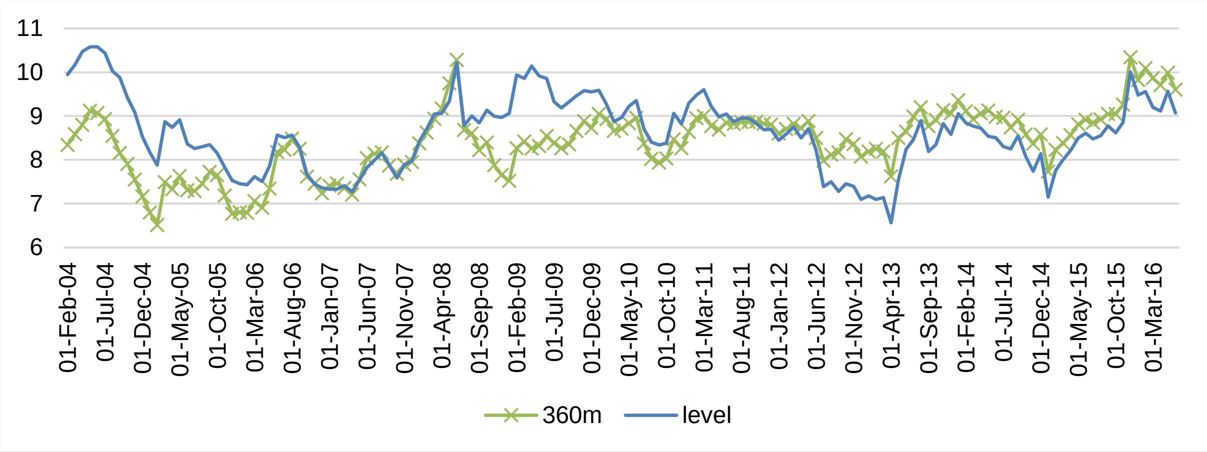


This leads to an additional method coined “the long term level” method (method 3). This method will proceed as follows:

Method 3: firstly, the estimation procedure described in method 1 for is performed and then the estimated level factor $\{\widehat{l_t}\}$ is extracted. Then this level factor is used in the place of the 360 month zero rate (the final observed rate). Afterwards, the estimation procedure is repeated for the whole term structure of interest rates.

An alternative method which makes use of the market segmentation hypothesis that states that different maturity buckets on the yield curve are determine independently by supply and demand. For that reason, when an attempt is made to extrapolate the zero curve, information obtained from the zero rates at short term maturities may not be relevant to explain the behaviour of the zero curve at long term maturities.

Figure 4.2: level vs 360 month spot rate



Hence an additional method is proposed that only takes into account zero rates at longer term maturities to fit the model and then use this to extrapolate the zero curve. This method is as follows:

Method 4: The factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}, \widehat{\lambda_t}\}$ are estimated by minimising $\mathcal{R}_f^2$ for the zero yield curve but only for zero rates at maturities longer than 60 months for the sample period February 2004 to June 2016. This will be referred to as long end fitting method.

Finally, an alternative method used by Thomas (2008), called the "final spot rate extrapolation" will be used. This method assumes that all spot rates that follow after the last observed spot rate maturity date, are equal to the last observed spot rate. Therefore, in this case, the zero rates after 120 months, i.e. 144 to 360 month maturities, are set equal to the 120 month observed zero rate. This method will be compared to the other three methods which use the Nelson & Siegel (1987) model for extrapolation purposes.

The performance of the above extrapolation methods is assessed at specific chosen dates corresponding to different zero yield curve shapes. Naturally, the benchmark for the performance is evaluated against the observed spot rates using calculating $\mathcal{R}_E^2$.

4.3 RESULTS

In Table 4.1, the estimation using the four methods are compared by evaluating the sum of fitted squared residuals ($\sum \mathcal{R}_f^2$) and the sum of extrapolated squared residuals ($\sum \mathcal{R}_E^2$) for the sample period February 2004 to June 2016. Varying λ provides the lowest fitted sum of squared residuals; whereas the long end fitting method provides the lowest extrapolated sum of squared residuals. Thus this method of extrapolating unobserved zero rates appears to be superior. The worst performing method is the final spot rate extrapolation method. Interestingly, constant λ and the final spot rate extrapolation only have a miniscule difference. The fact that the long end fitting method performed the best gives some strength to the market segmentation theory.

Table 4.1: Fitted and extrapolated squared residuals

Method:	$\sum \mathcal{R}_f^2$	$\sum \mathcal{R}_E^2$
Constant λ	15.88	204.17
varying λ	6.71	161.81
level as proxy for long term rate	N/A	185.09
Long end fitting	5810	126.17
final spot rate extrapolation	N/A	204.18

The values in Table 4.1 were determined for all curves over a relatively long period of February 2004 to June 2016. Therefore, for comprehensiveness, it is necessary to compare these methods at some selected dates. These dates were chosen to reflect the varying zero yield curve shapes.

In Figure 4.3, the zero yield curve for 30 September 2004 is plotted for (a) constant λ with $\mathcal{R}_E^2 = 4.716$. (b) varying λ with $\mathcal{R}_E^2 = 4.716$. (c) long term level factor with $\mathcal{R}_E^2 = 6.033$. (d) final spot rate extrapolation with $\mathcal{R}_E^2 = 2.574$. (e) long end fitting with $\mathcal{R}_E^2 = 2.980$. In this case the final spot rate extrapolation provides the best extrapolated rates compared to all the other three methods. Therefore, in this case, its concluded that using the Nelson & Siegel (1987) model fails to provide good extrapolated spot rates.

In Figure 4.4, the zero yield curve for 31 January 2005 is plotted for (a) constant λ with $\mathcal{R}_E^2$=3.395. (b) varying λ with $\mathcal{R}_E^2 = 4.876$. (c) long term level factor with $\mathcal{R}_E^2 = 4.132$. (d) final spot rate extrapolation with $\mathcal{R}_E^2 = 2.645$. (e) long end fitting with $\mathcal{R}_E^2 = 4.045$. Again, in this case, the final spot rate extrapolation method performs better than the other methods. It is apparent in both Figure 4.3 and 4.4 that the observed yield curve was downward sloping after the 120 month maturity date. This downward sloping behaviour of the spot rates is non-existent for maturities prior to the 120 month maturity. Therefore, it is not surprise that the Nelson & Siegel (1987) model, which used as inputs mostly observed upward sloping zero rates, poorly extrapolated.

Figure 4.3: 30 September 2004 extrapolated yield curve

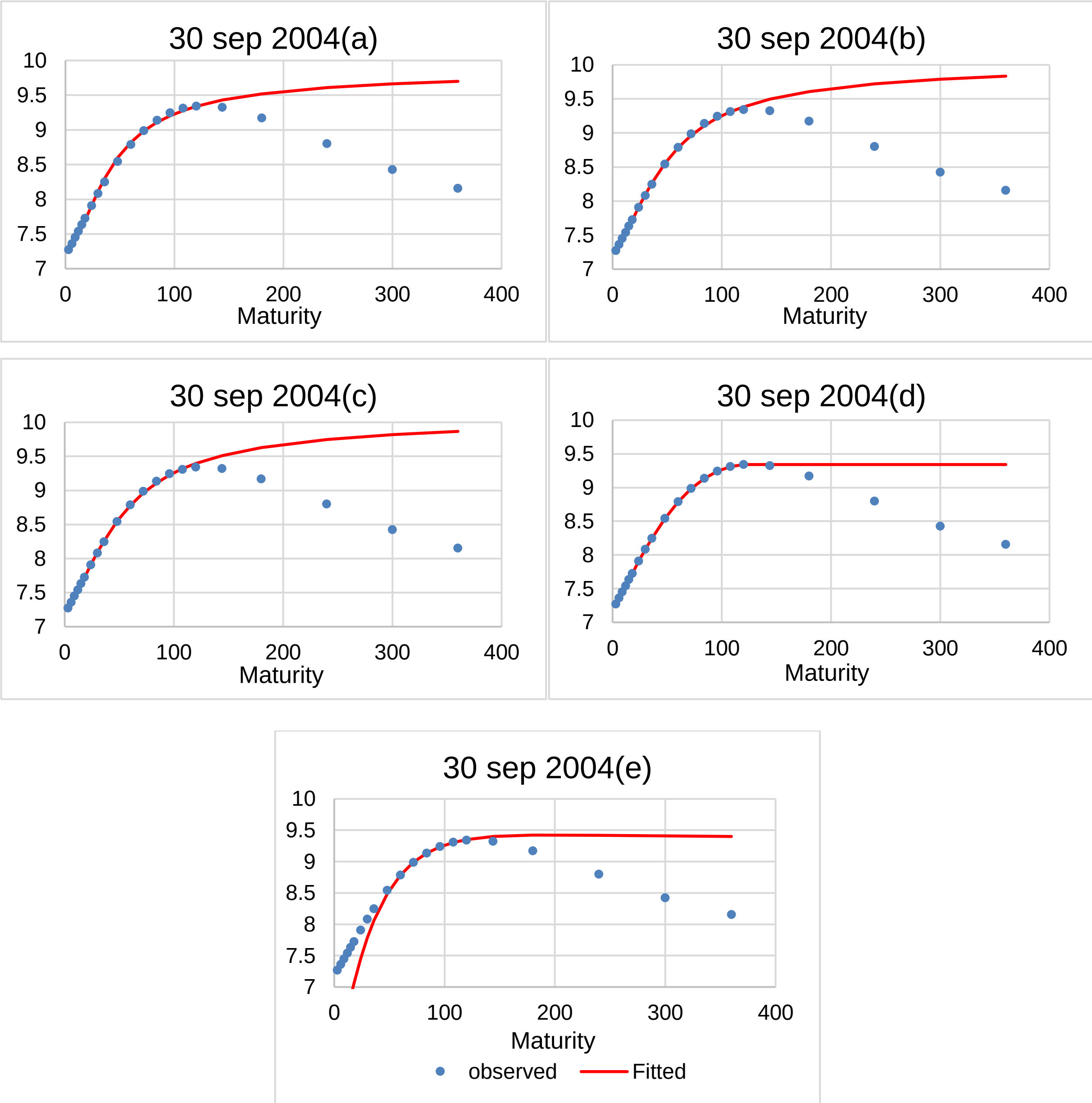
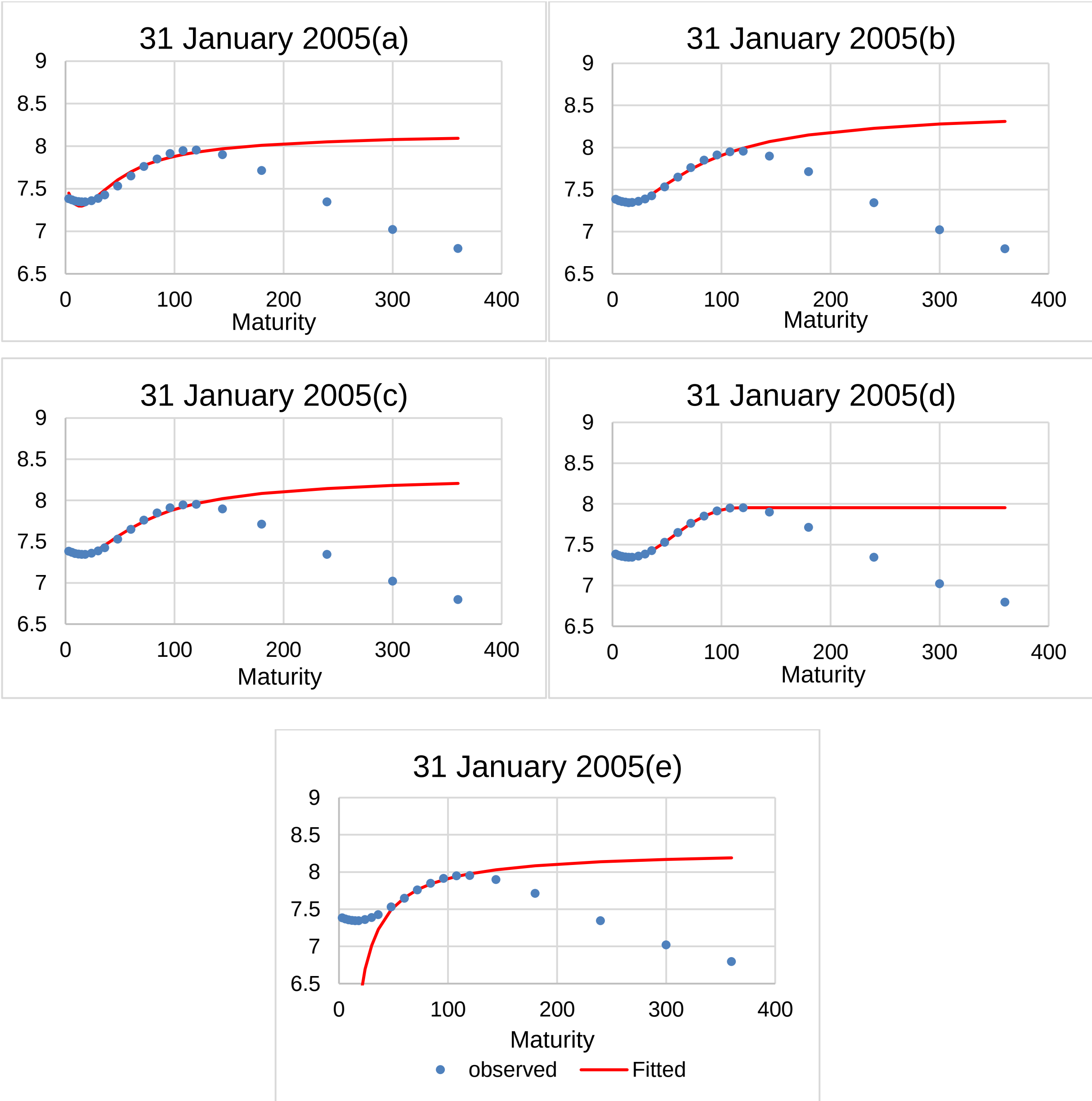
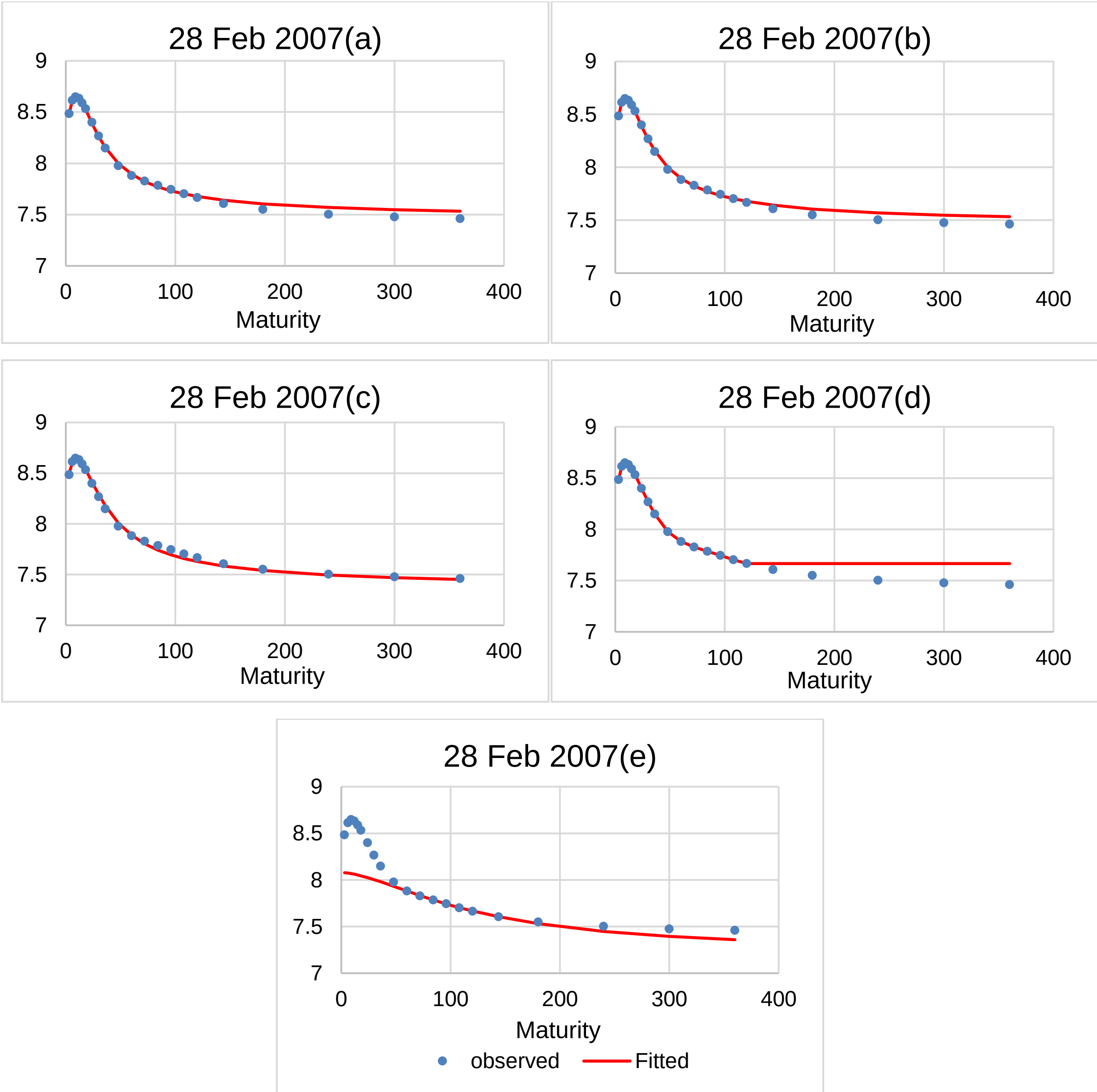


Figure 4.4: 31 January 2005 extrapolated yield curve



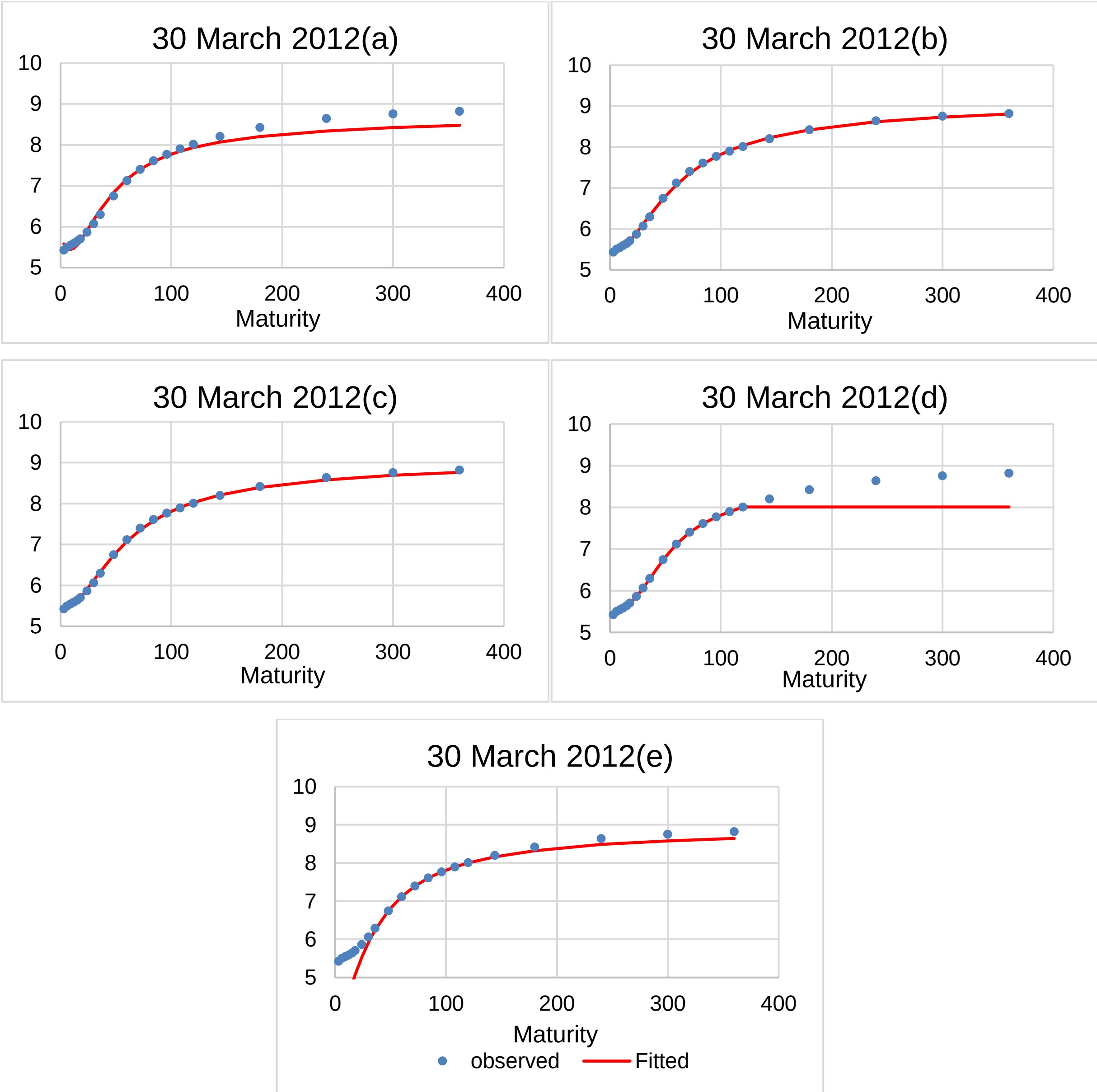
In Figure 4.5, the zero yield curve for 28 February 2007 is plotted for (a) constant λ with $\mathcal{R}_E^2 = 0.0025$. (b) varying λ with $\mathcal{R}_E^2 = 0.0187$. (c) long term level factor with $\mathcal{R}_E^2 = 0.0008$. (d) final spot rate extrapolation with the $\mathcal{R}_E^2 = 0.121$. (e) long end fitting with $\mathcal{R}_E^2 = 0.020$. In this case, the long term level method provides a very accurate fit for extrapolation and in contrast, the final spot rate extrapolation method performs the worst.

Figure 4.5: 28 February 2007 extrapolated yield curve



In Figure 4.6, the zero yield curve for 30 March 2012 is plotted for (a) constant λ with $\mathcal{R}_E^2 = 0.378$. (b) varying λ with $\mathcal{R}_E^2 = 0.002$. (c) long term level factor with $\mathcal{R}_E^2 = 0.011$. (d) final spot rate extrapolation with $\mathcal{R}_E^2 = 1.809$. (e) long end fitting with $\mathcal{R}_E^2 = 0.093$. In this case, the varying λ method provides the best extrapolated fit whereas the long term level method outperforms the constant λ method.

Figure 4.6: 30 March 2012 extrapolated yield curve



4.4 SUMMARY

To conclude, although the long end fitting method provided the best extrapolation results overall, there are instances where keeping λ fixed or using the method that replaced the long term yields with the level factor, provided a better fit. Hence, using the Nelson & Siegel (1987) method to extrapolate is limited because it only takes into account the fitted spot rates. Thus, to improve extrapolations, a method that not only takes into account the observed yields, but also other variables that can be used as proxies for the long term rates, will be required. One such approach would be to use an ultimate forward rate which provides a long term rate based on both economic variables and yield curve metrics such as the convexity and/or a term premia.

CHAPTER 5 CONCLUSION AND OPENING QUESTIONS

The Nelson & Siegel (1987) model is used to estimate the term structure of interest rates and since its introduction in 1987, a lot of research on it has been done. Although this paper is one of the numerous research papers that apply the model, it is only one of the few research papers that apply the model to the South African debt market. This thus provides additional insights into the model's applicability in emerging markets. As discussed in this paper, this model has numerous applications including fitting, time series forecasting and extrapolation of the zero yield curve.

In chapter 2, it was found that the Nelson & Siegel (1987) model greatly simplifies the task of modelling yield curves by representing the entire yield curve in terms of three unobservable factors. Since these factors are dynamic with time, autoregressive and vector autoregressive time series models can be fitted to them. This allows for fitting, forecasting and extrapolation of the zero rates. Furthermore, observable macroeconomic variables can be used to better the forecast if they are correlated with the latent factors. Finally, the entire term structure of interest rates sample period can be divided up into smaller samples periods in order to account for monetary policy regime changes.

In chapter 3, it was found that using the Nelson & Siegel (1987) model (under all assumptions considered) resulted in satisfactory zero spot rate fits. Additionally, over time, the estimated latent factors $\{\widehat{l_t}, \widehat{s_t}, \widehat{c_t}\}$ where extracted and then their interactions with macroeconomic variables were evaluated. Furthermore, an investigation into whether or not the estimated latent factors and the observed macroeconomic variables could be used to forecast the term structure resulted in inconclusive findings. This could be, in part, due to the static process applied to perform the forecasts. Therefore, more significant results could have been obtained if a more dynamic approach was applied. Also, the choices of macroeconomic variables used for this paper was limited; many more could have been used.

Finally, in chapter 4, the application of the Nelson & Siegel (1987) model to extrapolate the zero yield curve was performed. This analysis was performed on a reduced term structure, where only maturities up to 120 months where fitted. Thereafter, this was used to extrapolate the yield curve up to the 360 month maturity. This was done for all zero curves considered in the sample and also for the cases where the decay factor λ was fixed and varied. Extrapolation under the long end fitting method, which only considered zero rates after the 60 month maturity, performed better overall. However, after further investigation, this method was found to perform rather poorly for some zero curves. For such cases, additional adjustments, such as using some variable to approximate the long term rate, are required. For purposes of this paper, the extracted level factor was used as a proxy for the long term interest rate. Under certain circumstances, this adjustment outperformed the method of only using the Nelson & Siegel (1987) model to extrapolate.

Therefore, a more accurate approximation for the long term interest rates that includes observable macroeconomic variables (e.g. expected inflation) and term structure specific metrics such as term premia, should be investigated.

To conclude, this paper is limited in scope and further research is required into the modelling of the term structure over time and for extrapolation purposes in the South African context. An interesting venture will be an investigation of how extensions of the Nelson & Siegel (1987) model can be used for fitting and extrapolating the yield curve. One such extension, for example, is provided by Svensson (1994) who added an additional curvature factor to the Nelson & Siegel (1987) model. Moreover, this paper used the two step approach as described by Diebold & Rudebusch (2013). A more efficient estimation procedure that could be applied is the one step state space procedure, which uses the Kalman filter to estimate the yield curve latent factors. Therefore, a possible study might investigate whether this procedure is advantageous to the two step method in the South African context. Lastly, the no arbitrage assumption is not made in this study. Whether making this assumption could provide improvements of the forecasts and extrapolations of the South Africa's zero yield curve can also be investigated.

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