Calculation of Credit Value Adjustment using a series of Swaptions

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Abstract

The semi-analytical approach to calculating Credit Value Adjustment (CVA) on an interest rate swap (IRS) provides an alternative to the limited simplistic mark-to-market approach and the resource-intensive Monte Carlo approach. The semi-analytical (or swaption) approach was implemented in this paper using two models: Black's model and the Hull-White one-factor model. Results from using the semi-analytical approach were compared to that of the simplistic method. A relatively thorough analysis was also done on the factors affecting results from using the swaption approach, and the two models were also compared. It was found that the swaption approach provided far more information than the simplistic approach on how exposures are distributed over the two counterparties in an IRS and especially over the lifetime of the swap contract. Model parameters, as well as the term structure of interest rates, were found to have a significant effect on the CVA in each of the models. The Hull-White model, with an additional parameter over Black's model, showed more complex interactions between its parameters and CVA than Black's model did.

Opsomming

Die semi-analitiese benadering om die kredietwaarde aanpassing (KWA) op ‘n rentekoersuitruiltransaksie (RUT) te bereken is ‘n alternatief tot die beperkte, simplistiese “mark-to-market” benadering en die hulpbron-intensiewe Monte Carlo benadering. Die semi-analitiese (of “swaption”) benadering was geïmplimenteer met twee verskillende modelle: Black se model en die Hull-White een-faktor model. Resultate van die semi-analitiese benadering was vergelyk met dié van die simplistiese metode. ‘n Redelike deeglike analise was ook gedoen op die faktore wat die resultate van die semi-analitiese metode beïnvloed, en die twee modelle was ook vergelyk. Dit was vasgestel dat die semi-analitiese metode meer inligting as die simplistiese metode gee oor hoe krediet blootstellings in ‘n RUT tussen die twee teenpartye , en veral oor die leeftyd van die RUT, verdeel is. Dit was ook vasgestel dat model parameters, sowel as die rentekoers termyn struktuur, ‘n beduidende effek op die KWA het in elk van die modelle. Die Hull-White model, met ‘n ekstra parameter oor Black se model, het meer komplekse interaksies tussen sy parameters en KWA gehad as Black se model.

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List of common symbols used in formulas

The symbols listed here will have the same meaning throughout this paper.

$\boldsymbol{S(t,T,T_M)}$ – Value of a swap as seen at time t for a swap starting at time T and maturing at time T_M

$\boldsymbol{S_{t,K}}$ **–** Alternative notation for $S(t,T,T_M)$

$\boldsymbol{C(t,T,T_M)}$ – Swap rate as seen at time t for a swap starting at time T and maturing at time T_M. When dates are clear from context, only C will be used as notation.

$\boldsymbol{s_t}$ **-** Alternative notation specifically for $C(t,T,T_M)$

$\boldsymbol{D(t,T,T_M)}$ – Value at time T of a zero-coupon bond as seen at time t, maturing at time T_M with a principal of one unit of currency. Has different exact definitions under different models

$\boldsymbol{\lambda(T,T_M,Z^*)}$ - Re-parameterised, non-stochastic version of $D(0,T,T_M)$, for a specific value of random variable $Z = Z^*$

$\boldsymbol{f(t,s,T)}$ **–** Annual forward rate between time s and T, as seen at time t

$\boldsymbol{L}$ **–** Notional value of swap (or principal value of bond)

$\boldsymbol{h}$ – Length of the periods between payment dates. When written as $\boldsymbol{h_i}$, it denotes then the length of period i

$\boldsymbol{Rec}$ – Recovery rate

$\boldsymbol{K}$ **-** Strike rate (used as fixed rate when discussing tail swaps)

$\boldsymbol{EE(t)}$ – Expected exposure at time t

$\boldsymbol{\overline{S}(t,T,T_M)}$ **-** Value of a tail swap as seen at time t for a swap starting at time T and maturing at time T_M

$\boldsymbol{D_{tail}(t)}$ – Difference between normal swap and tail swap, when entered into at time t

$\boldsymbol{P(t,T,T_M)}$ – Price of a payer swap (any model)

$\boldsymbol{P_{HW}(t,T,T_M)}$ – Price of a payer swaption under Hull-White model

$\boldsymbol{P_{BS}(t,T,T_M)}$ – Price of a payer swaption under Black's model

$\boldsymbol{P_{HW\,tail}(t,T,T_M)}$ – Price of an option to enter into a payer tail swap under the Hull-White model

$\boldsymbol{P_{BS\,tail}(t,T,T_M)}$ – Price of an option to enter into a payer tail swap under Black's model

$\boldsymbol{R(t,T,T_M)}$, $\boldsymbol{R_{HW}(t,T,T_M)}$, $\boldsymbol{R_{BS}(t,T,T_M)}$, $\boldsymbol{R_{HW\,tail}(t,T,T_M)}$, $\boldsymbol{R_{BS\,tail}(t,T,T_M)}$ - Same as above, except for receiver swaptions/swaps

$\boldsymbol{\xi(t)}$ – Value of numeraire at time t

$\boldsymbol{N(Z)}$ – Standard normal cumulative distribution function

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List of commonly used abbreviations

CVA – Credit Value Adjustment

DVA – Debit Value Adjustment

NPV – Net Present Value

EE – Expected Exposure

IRS – Interest Rate Swap

IFRS – International Financial Reporting Standards

GAAP – Generally Accepted Accounting Principles

CSA – Credit Support Annex

OTC – Over The Counter

MtM – Mark-to-Market

1 Introduction

The financial crisis of 2007 has prompted firms and economic agents to re-examine their approaches to assessing financial risks, with counterparty risk comprising a large part of these risks globally. In the over the counter (OTC) derivatives market, counterparty risk has gained immense attention in recent years.

OTC derivatives comprise a large part of the global financial market. In December 2008, the overall outstanding notional in the OTC market amounted to $547 trillion, with 70% of that consisting of interest rate derivatives. Even though the financial crisis had been a recent occurrence, with the global economy still recovering, the OTC markets' gross value increased by 60% from June 2008 to December 2008, while specifically interest rate derivatives' gross market value doubled from $9 trillion to $18 trillion in that same period (Stein and Lee, 2010:2). With the absence of a mediator (e.g. a market-maker) in the OTC derivatives market, counterparty risk has always been a real danger, even though it has not always been recognised as such.

Counterparty risk is the risk that the counterparty in a derivative (or any financial) contract will fail to meet its obligations, possibly leading to losses for the party that was defaulted on. Where the risk cannot be eliminated or significantly reduced, e.g. with a Credit Support Annex (CSA), a fair value adjustment is made to both the parties' derivative books to make provision for the credit risk. This adjustment is known as the credit valuation adjustment, or CVA. Pykhtin and Zhu (2007:1) define CVA as the price of counterparty credit risk. Accounting standards (IFRS and GAAP) require that credit risk be reflected in the fair value measurement of derivatives (Shearman and Sterling LLP, 2013:2).

1.1 Problem statement

Before the global financial crisis, firms limited their counterparty risk by trading only with financially secure counterparties. However, during the financial crisis, even these "secure" counterparties were shown to pose financial risks, sometimes even the biggest (Gregory, 2012:18). In response to this, Basel implemented a requirement that a fair value adjustment (CVA) be made to each derivative contract in the OTC market (Shearman and Sterling LLP, 2013:1).

CVA can be calculated using a variety of methods, with the Monte Carlo simulation approach rapidly becoming a standard because of its flexibility in incorporating various risk factors (Gregory, 2012:157). However, the focus of this paper is solely the semi-analytical method (or swaption approach). If the Monte Carlo approach is used, the estimate will generally be very flexible and accurate regarding risk factors, but the infrastructure (in terms of technology and human capital) and time required for this approach is sometimes impractical for certain firms.

Firms that lack infrastructure can instead use the semi-analytical approach to estimate expected exposure (EE), which will then be used to calculate CVA. This approach sacrifices some flexibility in estimating the exposure and can only be applied to a portfolio containing a relatively simple combination of risk factors. However, it is usually quick and easy to implement compared to the Monte Carlo method (Gregory, 2012:159). This will allow firms with less infrastructure to estimate CVA with more accuracy, without sacrificing large amounts of resources to do so, should the semi-analytical method prove to be significantly more accurate than the simplistic method. It therefore needs to be determined whether there is indeed any merit for choosing the semi-analytical method over the simplistic approach.

1.2 Research objectives

In this research assignment, the effectiveness of using the semi-analytical method in the calculation of EE of interest rate swaps will be investigated. The CVA will then be calculated on a relatively small and simple portfolio of interest rate swaps and results will be compared to the mark-to-market method. In addition, the following areas will be explored and discussed:

- Assessment of interest rate risk after entering into a swap, using the DV01 measure
- The effect of collateral on the exposure profile as well as CVA
- Calculation of exposure and CVA for a simple netted portfolio
- The effects that model parameters have on EE and CVA valuation
- Comparison of different interest rate models in EE and CVA valuation
- Calibration of the swaption models to South African market data, specifically forward interest rate curves (using JIBAR rates)
- Investigate/mention any other issues that might come up during research, as well as identifying areas that might require further research

1.3 Clarification of key concepts

A short definition will be given here for common terms used in this paper.

Credit value adjustment (CVA)

Defined by the Basel III committee as the difference between the prices of a derivative contract if the counterparty is risk-free and if the counterparty poses default risk (Shearman and Sterling LLP, 2013). Thus, it is the adjustment to the fair value of a derivative contract (in this case, an interest rate swap) to account for the risk that a counterparty bears or represents.

Debit value adjustment (DVA)

Has the same definition as CVA, except that the party's own riskiness in the contract is measured, thus it measures CVA from the counterparty's perspective. It can be defined as the CVA that a counterparty would be expected to hold against the party concerned (International Valuation Standards Council, 2013:6).

Expected exposure (EE)

Defined as the average of the distribution of all exposures on a specific date in the future, conditional on positive market values (Harper, s.a.). In simpler terms, EE is the expected loss that will be incurred at a future point in time if a counterparty defaults. This is based on the assumption that the party that was defaulted on will replace the swap that was lost due to default, at the same instant that default occurred, and the loss will be the amount that the party will have to pay to enter into the new swap.

Interest rate swap (IRS)

A derivative contract that lets two parties exchange fixed and floating rate payments. In other words, one party pays a fixed rate and receives a floating rate (usually based on a set reference rate, such as JIBAR) to and from a counterparty, and vice versa. A swap where the holding party pays fixed and receives floating will be referred to as a payer swap and a swap where the holding party receives fixed and pays floating, as a receiver swap.

Swap rate

The fixed rate in an interest rate swap. Theoretically, this is determined by setting the IRS value to zero at inception.

Swaption

A swaption, also known as an option on an IRS, is the right, but not the obligation, to enter into a swap contract at a predetermined swap rate. Swaptions can either be i) European, in which case the only exercise date available is at the expiry of the option, ii) American, where exercise is possible on any date for a limited time span, or iii) Bermudan, where several predetermined dates for possible exercise are laid out for a limited time span.

A swaption has the following components

- An underlying swap, which the holder will enter into upon exercise.
- Strike rate, which is the swap rate of the underlying swap.
- Maturity date, after which the swaption has expired and cannot be exercised anymore.

Short rate

The short rate is the prevailing interest rate for an infinitesimally short period of time. Interest rate models that are used in this paper to price swaptions, describe the evolution of this rate.

Term structure models

Hull (2012) defines term structure models as interest rate models that describe the evolution of all zero-coupon interest rates. Hull further divides term structure models into equilibrium models and no-arbitrage models.

In an equilibrium model, assumptions are initially made about economic variables. These specify the parameters of the model, which will in turn give bond and option prices, but most importantly a term structure as output. Examples of equilibrium models are the Cox, Ingersoll, Ross (CIR) model, and the Vasicek model.

No-arbitrage models improve on equilibrium models by providing an exact fit to the term structure. This is done by using today's term structure of interest rates as input, instead of an output. Examples are the Hull-White one and two-factor models and the Ho-Lee model.

Tail swap

Stein and Lee (2010:11) define a tail swap as an IRS where if the swap is entered into between two payment dates, the full cash flow for that period is paid and received as if the swap was entered into at the beginning of that period. This is in contrast with a normal interest rate swap, where the first cash flow is prorated.

Recovery rate

Assume party A defaults while indebted to B. The recovery rate is the percentage of assets owed to B, that B would be able to recover.

Probability of default

The probability that a counterparty will default on the contract. It is expressed as a probability distribution over time. In this paper, it will be assumed as a constant.

Numeraire

The valuable good in which all prices in an economic model are denominated (Reis and Watson, 2007:1). Realistically, this will be the money market account, but a riskless zero-coupon bond with a specified maturity is also a common numeraire (in mathematical models).

DV01

Hull (2012:804) defines DV01 as the dollar value of a 1-basis-point increase in all interest rates (parallel shift in yield curve), although some sources define it for a 1-basis-point decrease, rather than an increase. To clarify, it reveals by how much an instrument's value will change in currency units for a basis point shift in the yield curve. In this paper, it will be calculated for a basis point decrease.

Semi-analytical method

A method for calculating CVA that views exposure over time as a series of swaptions. Considered less accurate, but quicker to implement than Monte Carlo methods.

Simplistic method

A method for calculating CVA that views exposure at the present as the (present) market value of the swap.

Mark to Market Accounting

Harvey (2011) defines mark to market accounting as referring to "accounting for the fair value of an asset or liability based on the current market price instead of book value." The method for determining the market price of swaps will be treated in the literature review in section 2.2.1.

Over the Counter (OTC)

Refers to derivative contracts that are customised between two counterparties. This is in contrast to exchange traded derivatives.

1.4 Importance and benefits of the study

As already noted, the semi-analytical method requires less time to implement than the Monte Carlo method. Firms that need to calculate CVA on a continual basis, but do not have a lot of time or resources for the process, will therefore find the semi-analytical beneficial. The approach has less accuracy and flexibility compared to the Monte Carlo method, but is much faster to implement and requires less resources. This allows firms with fewer infrastructures to estimate CVA with more accuracy, without sacrificing large amounts of resources to do so.

Chapter outlines

The reader has been made aware in this chapter of the motivation for this research assignment, as well as the basic plan for this paper. The reader has also been prepared for the following chapters by the definitions given here. In the following chapter, a detailed literature review will be provided regarding the theoretical foundations of the study. In chapter 3, the specific methodology employed in the study will be given, and in chapter 4 will be followed by a comparison of the results of the tests performed. This paper will conclude by stating the limitations of the study as well as discussing some areas for future research.

2 Literature review

This literature review will cover all the technicalities necessary to understand the semi-analytical approach from a mathematical point of view. It begins by explaining the basic workings and pricing of IRS contracts, along with standard tools for mitigating credit risks inherent in swaps. CVA is then discussed, first at a basic level, after which a very detailed discussion is given on how it can be calculated under specific models, as well as under certain agreements, namely netting and collateral. A guideline on how to calculate CVA DV01 is also given under each model.

2.1 Swap contracts

An IRS contract, (which will be referred to from here on as a swap) is defined by Ferrara and Nezzamoddini (2013:6) as a bilateral contract between two parties to exchange cash flows based on different indices at periodic dates in the future. Each contract has a notional amount that determines the actual payments, normally by multiplying the notional by the specified rates. A plain vanilla swap, which is the type mostly discussed in this paper, is a contract to exchange fixed payments for floating payments at periodic dates in the future. These periodic dates are usually monthly, quarterly, semi-annually or annually. The periodicity of the two payment streams can also differ, for example, fixed payments might be quarterly, while floating payments are semi-annual.

The exchange of payments will be achieved by one party's (party A) agreement to pay the counterparty (party B) a fixed rate on the notional, periodically, while the counterparty agrees to pay party A, a floating (that is, variable) rate. This rate is normally determined by some publicly quoted rate, such as the LIBOR rate, or, in South Africa, the JIBAR rate.

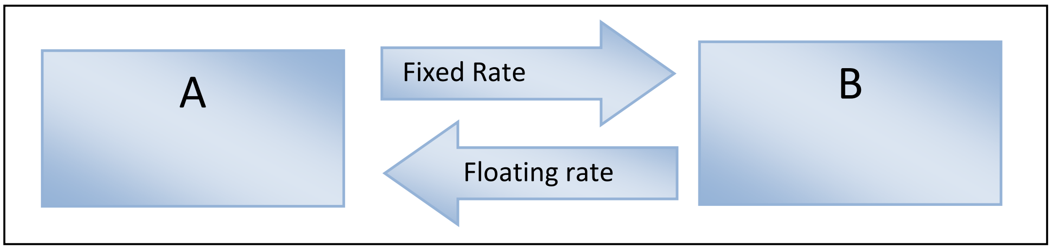


Figure 2-1: Payments exchanged on a payment date in a vanilla swap

There are various possible motivations for entering into a swap contract: Party A, for example, might have a loan that it is currently paying JIBAR on, hence it suffers the risk of increasing interest rates. To hedge against this risk, party A could enter into a contract similar to the one described above. This means that A will receive floating rate payments, based on the same notional and index that its loan's interest is calculated on, with which it can make payments on its loan, and in return pay B a fixed rate. This would have effectively transformed party A's floating rate loan into a fixed rate loan. Party B, in this case, would normally operate under the belief that interest rates will fall after the swap was entered in, since B is now paying the floating rate.

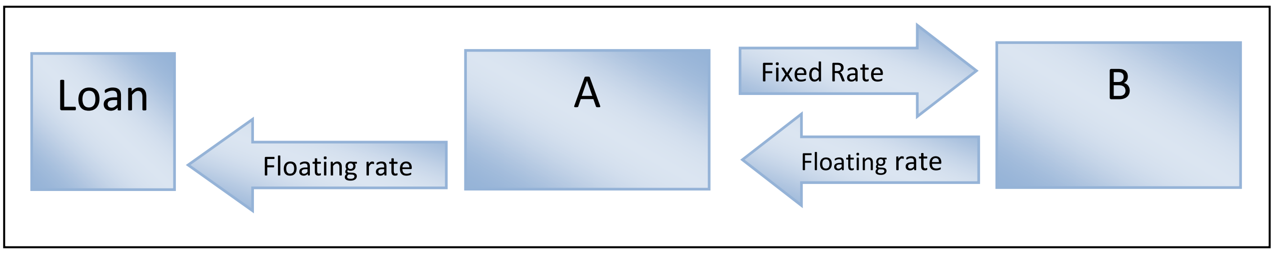


Figure 2-2: Payments exchanged if party A transforms a loan

2.1.1 The pricing of swaps (simplistic)

Ferrara and Nezzamoddini (2013:6-7) states that the market value of a swap can be viewed as the difference in value between two risk-free bonds – one paying a fixed rate, the other a floating rate. For the party receiving fixed, this can be expressed mathematically as

$$\boldsymbol{S(0,0,n) = B_{fixed} - B_{floating}} \qquad \textbf{(1)}$$

where

- $S(0,0,n)$ is the current value of the swap that is at inception or already entered into, and that matures in n years.
- B_{fixed} is the value of the fixed-rate bond at time 0
- $B_{floating}$ is the value of the floating rate bond at time 0

and,

$$\boldsymbol{B_{fixed} = F_{fixed} \sum_{i=1}^{n} \frac{C(0,0,n).h}{(1+r_i)^i} + \frac{F_{fixed}}{(1+\ r_n)^n}} \qquad \textbf{(2)}$$

$$\boldsymbol{B_{floating} = \sum_{i=1}^{n} \frac{f(0,(i-1)h,ih)h}{(1+r_i)^i} + \frac{F_{floating}}{(1+\ r_n)^n}} \qquad \textbf{(3)}$$

assuming that there are n payments in total to be exchanged during the swap contract, and that payments have the same periodicity, and where

- h is the length of each period in years (assuming all periods have the same length).
- $C(0,0,n)$ is the current market swap rate for an IRS that matures n years from the present, or alternatively here, the fixed bond coupon rate.
- F_{fixed} and $F_{floating}$ are the face values for the fixed and floating rate bonds, respectively.
- $f(0,(i-1)h\,,ih)$ is the annual forward interest rate between time $(i-1)h$ and time ih
- r_i is the required rate of return for a period of ih, where i is the number of coupon periods up until that point in time, and h is the length of each coupon period.
- n is the number of periods until the bonds mature.

In the simplistic view of swap rate pricing, the value of the swap contract is assumed to be zero at inception of the contract. This view also does not take into account CVA or DVA, thus it is assumed that each counterparty has a default risk of zero. The swap rate will be the coupon rate that the fixed value bond pays which will cause the value of the swap to be zero at inception, so

$$S(0,0,n) = B_{fixed} - B_{floating}$$

becomes

$$\mathbf{0} = \boldsymbol{B_{fixed}} - \boldsymbol{B_{floating}} \quad \textbf{(4)}$$

Using the definitions for the fixed and floating rate bond, equations (1) and (2), with equation (3), a formula for the swap rate can be derived as

$$\boldsymbol{C(0,0,n)} = \frac{\mathbf{1} - \boldsymbol{D(0,0,n)}}{\sum_{i=1}^{mn} \boldsymbol{D(0,0,\frac{i}{m})h}} \quad \textbf{(5)}$$

where

- n is the number of years, from inception of the swap, until the swap matures.
- m is the number of payments (or periods) per year.

The full derivation is given in Appendix A.

Once again, note that these formulas describe a simplistic approach to pricing swaps, that is, without taking into account counterparty risk. In the real world, however, there normally exists counterparty risk, however small. Therefore the subsequent, necessary adjustment to the value of a swap to account for this riskiness, which is known as CVA and DVA (defined in section 1.3), is still required.

2.1.2 Mitigation of credit risk

Although there are many ways to mitigate counterparty risk (Gregory, 2012:41), only the two most common ways of doing so - netting and collateralisation - will be discussed and investigated in this paper.

2.1.2.1 Netting

Gregory (2012:46) divides netting into two categories: Payment netting and closeout netting.

Payment netting

Payment netting lets an institution combine its regular cash flows (daily, weekly, monthly, etc.) occurring on the same date with a certain counterparty, into one cash flow on that date. If, for example, party A owes party B R5 million and B owes A R6 million on a certain day, B would simply pay A the net value of those two amounts, which is R1 million. This increases operational efficiency, since now only one payment is made between the parties, instead of many, but also reduces risk in case of unexpected default by one of the parties.

If payments were not netted between party A and B, and party A payed B R5 million on the morning of a certain day, expecting the payment from B during the course of the day, and B defaults later that day before paying A the amount due, A would lose the R6 million that it was entitled to. In comparison, if payments were netted A would have lost only R1 million.

In a presentation by Pallavicini (2010:44-45), three simple cases for netting were considered. The first case is a portfolio of swaps consisting of swaps that are entered into on the same date, but that have different maturities. The netted portfolio is then equal to an amortising swap with decreasing principal. Diagrammatically, this portfolio is represented as follows

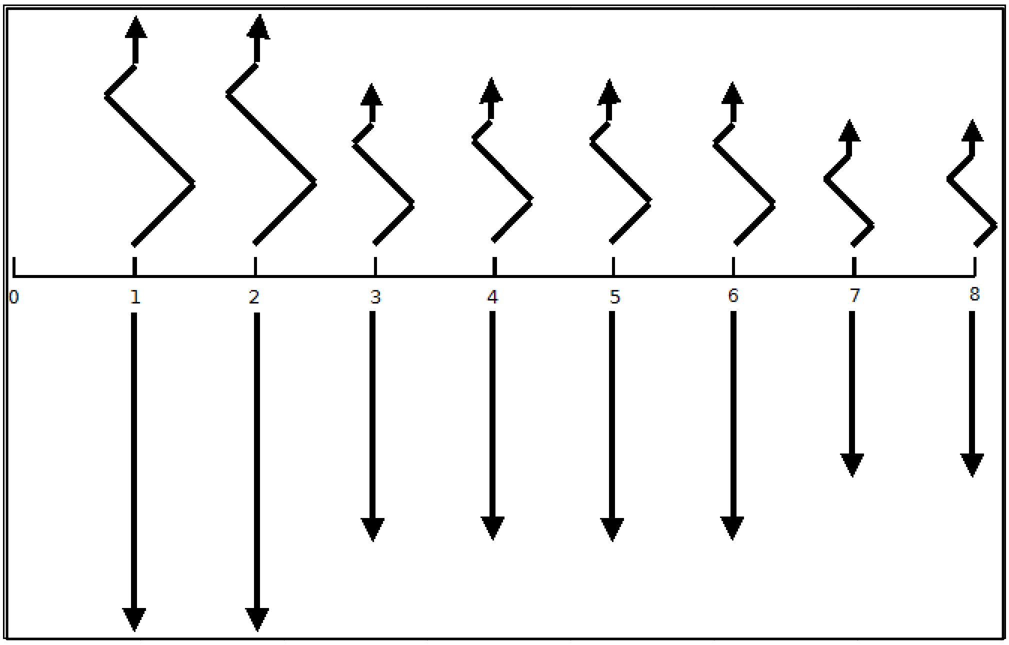


Figure 2-3: Cash flows for a portfolio that consists of 3 swaps at time zero, with one swap maturing at time 2, another at time 6, and the last at time 8

Note that in the diagram above, and in the following two diagrams, incoming fixed cash flows are represented by downward pointing straight arrows, and outgoing floating cash flows are represented by upward pointing zigzag arrows. All the swaps in the portfolio are therefore receiver swaps. All cash flow diagrams with timelines will be represented in this way from here on.

The second case is a portfolio of swaps that have the same maturity date, but different starting dates. This netted portfolio is equivalent to an amortising swap with increasing principal. In diagram form, it can be represented as

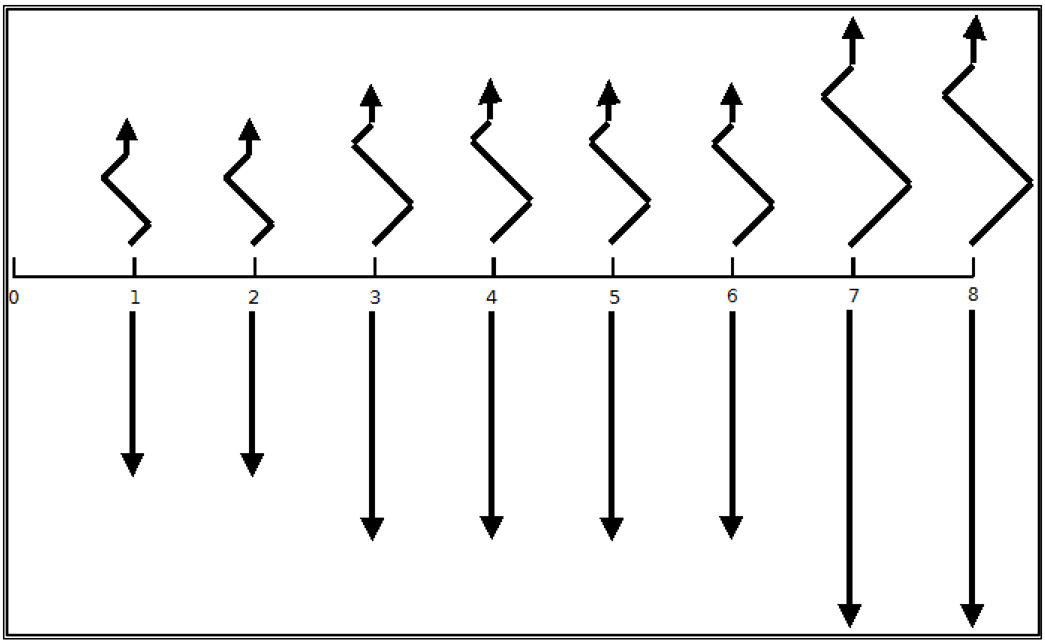


Figure 2-4: Cash flows for a portfolio that consists of 1 swap just after time zero, enters into another at time 2 and another at time 6

The third case is a portfolio of swaps that have the same starting and maturity dates. This is equivalent to a swap with a principal amount equal to the sum of the principal amounts of all the swaps in the netting agreement, and with the same starting and maturity dates. In diagram form, it looks as follows

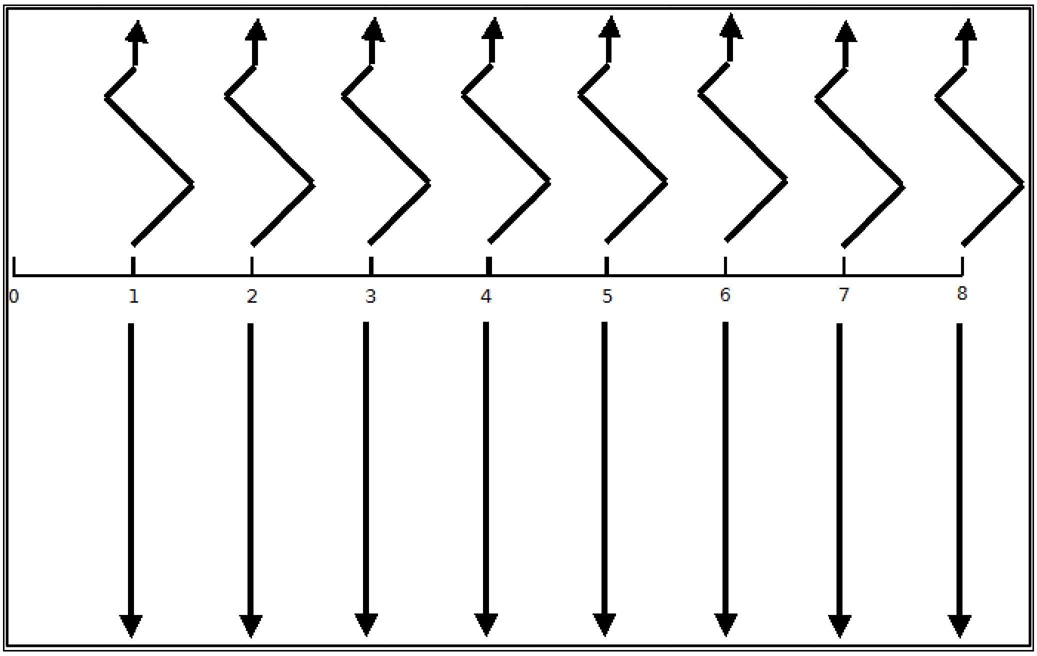


Figure 2-5: Cash flows for a portfolio that consists of a number of swaps at time zero, and does not add or sell, or have any swaps maturing before time 8

The three situations described above also apply to a portfolio of payer swaps.

Closeout netting

Closeout netting occurs when a counterparty defaults. Gregory (2012:48) further divides closeout netting into two parts: Closeout, which is the right of the party being defaulted on to cease any contractual obligations with the counterparty, and netting, which is the transferral of the net amount of positive and negative balances remaining in the contracts. This allows the party being defaulted on (party A) to recover the net value of all the current contracts with the defaulting counterparty (party B). If A owed B the net amount, A would make the payment, but if A was owed the net amount, A would make a bankruptcy claim on the amount.

The alternative, without netting, would have been for party A to pay party B the amount that was owed to B, then make a bankruptcy claim on that amount, which would in most cases be a much bigger amount than in the case of closeout netting, and thus the recovery rate would also be much lower for A. It is easy to see that closeout netting has a direct effect on CVA, since CVA is directly proportional to the recovery rate, so therefore the presence of closeout netting would reduce CVA significantly.

2.1.2.2 Collateralisation

Gregory (2012:59) defines collateral as an asset supporting a risk in a legally enforceable way. The concept behind collateral is simple: if the exposure on an OTC derivative contract exceeds a certain amount (threshold amount), then the counterparty (party A) to whom the derivate has a negative net value has to post a certain agreed-upon amount. This amount will be collected by party B, the other party in the contract, in case A defaults.

Collateralisation has clear benefits, as pointed out by Gregory (2012:60). Firstly, it limits credit exposures with counterparties, enabling institutions to do more business, keeping credit exposures within credit limits. It also allows institutions to do business with a wider variety of counterparties, as some institutions might, as matter of policy, not be allowed to do business with certain parties that fall below a certain credit rating. Collateralisation eliminates many of the inherent risks of dealing with low-rated counterparties. Since collateralisation will effectively reduce CVA, it will also allow derivative contracts to have more competitive prices. Finally, it will reduce capital requirements for institutions desiring to enter into certain derivative contracts.

Generally, collateral agreements also have the following considerations:

- Threshold amount: the required minimum exposure before collateral is posted, i.e. how much exposure each counterparty is prepared to accept.
- Margin Call: a request by the party to whom the contract has net positive value, to the counterparty (with negative net present value) to post additional collateral.
- Remargin period, or the frequency of margin calls: Generally the frequency at which CVA is calculated (e.g. daily, weekly, etc.).
- Independent amount: amount that is posted over and above collateral, to account for additional risks, mainly the risk connected to the remargin period.
- Credit support amount: total amount of collateral that is posted by a counterparty on a certain date.
- Grace period: the amount of days before a defaulted counterparty's position will be finally closed or liquidated, measured from the date of default.

To simplify the analysis, though, the grace period and independent amount will be assumed as zero.

Bringing threshold amount into the posting of collateral, the credit support amount on any date t can be represented by the formula

$$\boldsymbol{Coll(t) = Ex(t) + I - H}$$

or

$$\boldsymbol{Coll(t) = Ex(t) - H} \tag{6}$$

Where

- $Coll(t)$ is the credit support amount (or collateral) on date t
- $Ex(t)$ is the exposure on date t
- I is the independent amount
- H is the threshold amount agreed upon by both parties at the signing of the contract

Note that this is a simplified version, as other amounts could also be added to the credit support amount.

Collateralisation also changes the way a swap is valued. Since the amount that is posted is held by the party to whom the swap has a positive value, it will earn interest on that amount for the period that the collateral amount is held. This affects the pricing of the swap. However, it is a relatively complicated matter to make this adjustment, with most of the available literature on this matter theoretical, rather than practical. More importantly, it is not the main focus of this paper to value swaps, so therefore this adjustment will not be made when calculating current market values for swaps under collateral agreements. More on this topic can be found in a paper by Johannes and Sundaresan (2002).

2.2 The swaption approach to calculating CVA

Since swaps exhibit bilateral characteristics, the exposure resulting from a position in an interest rate swap can be represented as a series of swaptions. This particular approach was first formalised in a paper by Sorensen and Bollier (1994:24). They interpreted each swaption as the cost of replacing the swap that was lost (because of a counterparty's default). This is the discounted expected exposure (EE) for the time of the possible default, at the current time. Given that two parties, A and B, enter into an interest rate swap, they represent the adjustment from A's point of view as follows

$$CR_A = PD_B \, EE_A$$

where

- CR_A is the credit risk adjustment allocated to B's risk of default
- PD_B is the probability that B will default on the single default date
- EE_A is the value of the option for A to replace the swap

The formula above assumes that there is only one possible default date. Although greatly simplified, and written long before the global financial crisis of 2007 (which challenged many of the assumptions of credit risk previously held) the expression above communicates the essence of the swaption approach to calculating CVA. That is, that CVA is essentially the exposure multiplied (or weighted) by probability of default on that date. Note that in reality, CR_A is calculated for each possible default date, which includes dates between coupon payment dates, then integrated to find the final value of CVA. These specifics, however, will be handled in section 2.2.1.

To illustrate the parallel between exposure and a swaption, consider the following: Let the swap value, defined by S, have a negative value for party A at the time of party B's default (a liability). Pykhtin and Zhu (2007:1) notes that it is convenient assume that the party that was defaulted on (A in this case) will enter into a similar contract with a different counterparty to maintain its position. A would therefore close out its position by paying B the market value of the contract, and receive the same amount upon entering into a similar contract with a third party, C. Therefore in this case, A's net gain/loss would be zero. This is illustrated by Figure 2-6 below.

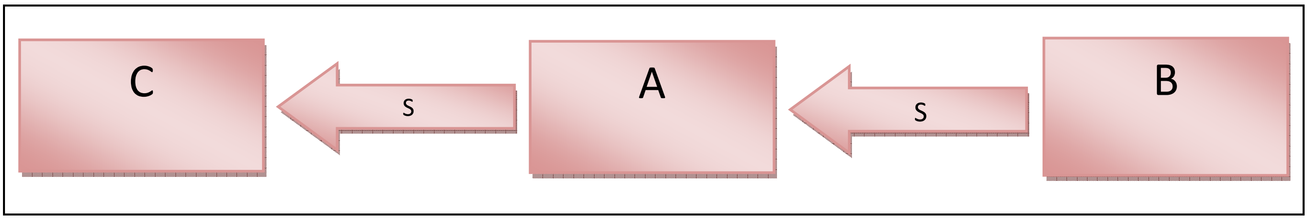


Figure 2-6: The case of the swap having a negative net value for A, if B defaults. Financially, the default event does not affect A.

On the other hand, if the swap (S) had a positive value for A at the time of B's default, meaning that the swap was an asset to A at that time, A would close out its position with B, receiving nothing (since B is defaulting) and enter into a similar contract with C, paying the market value of the contract, and suffer a net loss. This is described by Figure 2-7 below.

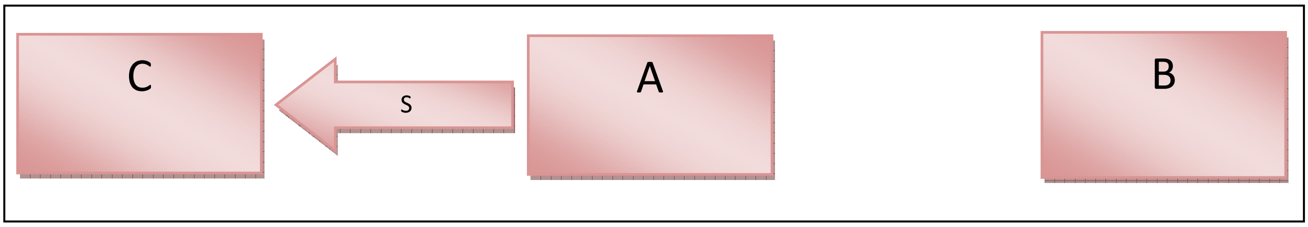


Figure 2-7: The case of the swap having a positive net value for A, if B defaults. Financially, the default event affects A negatively.

A's exposure at any time can therefore be modelled as max(S_A, 0), where S_A is the market value of the swap that A has entered into, at the time of default. This is the formula for the value of a European Swaption at expiry.

The example above is only for unilateral counterparty risk though, meaning that it takes into account only B's possibility of default. In reality, normally both parties' probability of default has to be taken into account. For this, Sorensen and Bollier (1994:28) suggest that a bilateral credit model has to be built. Suppose that A and B enter into an interest rate swap. Both parties have option positions that will create risk in both directions, both have credit risk, and both estimate future times at which their position will be an asset, and times at which their position will be a liability. Party A will estimate its credit risk as

$$CR_A = PD_B\, EE_A - PD_A\, EE_B$$

where the variables have the same interpretation as in the unilateral approach. Note that the value $PD_A\, EE_B$ represents the DVA (see section 1.3) from party A's point of view. The formula above is in accordance with McGlinchey (2014), who states that bilateral CVA is the netted values of unilateral CVA and DVA. Since this paper focuses solely on the CVA, only a unilateral model will be built.

When using the semi-analytical approach, Gregory (2012:158-159) states that simplifying assumptions regarding the risk factors behind the exposure have to be made. This is the main reason for the semi-analytical approach being an approximation, rather than a more exact estimate. Gregory also states that (i) path-dependent factors, such as early exercise decisions, are hard to reflect in the model, (ii) when collateral is present, the formulae in the analytical approach have to be altered to account for this, and (iii) only simple cases of netting can be incorporated in order for the model to still have its simplifying advantages (see section 2.1.2).

2.2.1 Some specifics

Sorensen and Bollier (1994), although being the first recorded to mention the idea of modelling EE on a certain date as a swaption, only presented a simplified version of the idea. A more practical treatment is given here.

Once expected exposures for a satisfactory number of future dates have been calculated, the required CVA can then be calculated. CVA will give a type of summary to all the expected future exposures, weighting each figure according to the probability of default at that particular time.

Stein and Lee (2010:10) express CVA as

$$\boldsymbol{CVA = \int_0^T (1 - Rec) P_{tail}(0, t, T) p(t) dt} \qquad \textbf{(7)}$$

where

- $P_{tail}(0, t, T)$ is the value at time 0 of an option to enter into the tail of a payer swap at time t, that matures at time T. This value also represents the EE. If CVA for a receiver swap was to be calculated, $R_{tail}(0, t, T)$ would have been used instead.
- $p(t)$ is the probability of default density at time t
    - Note that for this paper, simplistic recovery rates and probability of default will be assumed, hence probability of default will also be considered independent of $P_{tail}(0, t, T)$
- Rec is the recovery rate, which we will define as zero (as it is out of the scope of this research assignment)

Gregory (2012:243) defines a discrete formula for CVA as

$$\boldsymbol{CVA \approx (1 - Rec) \sum_{i=1}^{m} D(0, 0, t_i)\, EE(t_i) PD(t_{i-1}, t_i)}$$

where

- $EE(t_i)$ is the expected exposure at time t_i (the adjusted swaption value)
- $PD(t_{i-1}, t_i)$ is the probability of default between time t_{i-1} and t_i
- $D(0, 0, t_i)$ is the value at time 0 of a zero-coupon bond, as seen at time 0, that matures at time t_i. This will be referred to as the discount rate.

A full proof for formula (7) is given in Appendix A. Formula (7) implies that CVA can be seen as the area under the plot of probability of default times EE, on the y-axis, against time, on the x-axis.

2.2.2 A few subtleties

In the case of modelling financial exposure to default on a swap contract, a slight adjustment has to be made to the underlying swap. In the event of default between two payment dates, say, a default on date t, between dates t_1 and t_2 as shown below, a normal swaption will take into account only the period between t and t_2. Mathematically, the fixed rate payment is scaled according to the shortened period in relation to the full period, while the floating rate payment will be calculated on a shortened index between t and t_2. The first payment is therefore prorated (Stein and Lee, 2010:11).

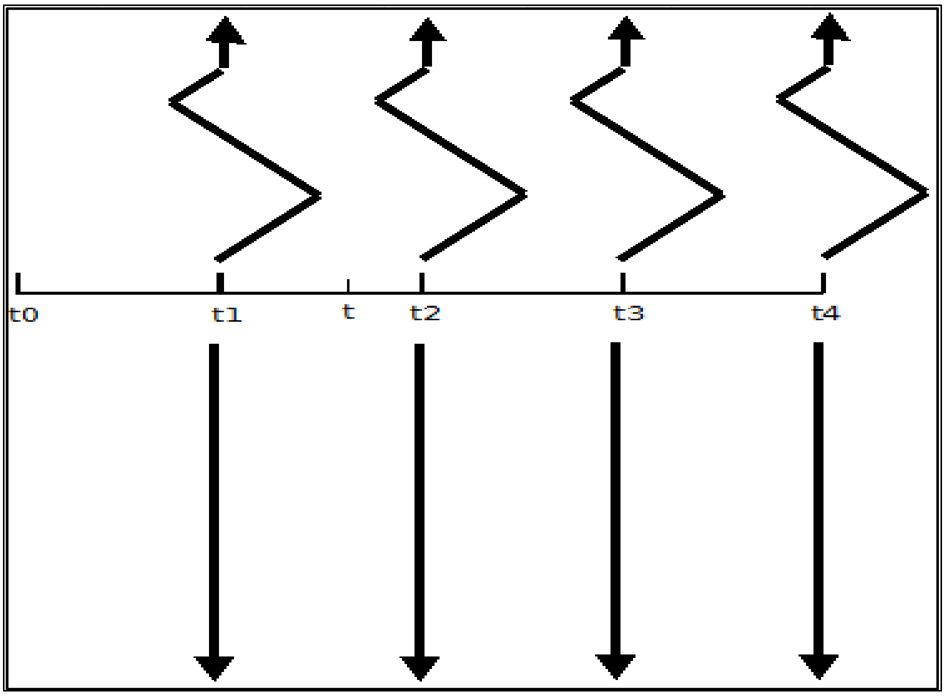


Figure 2-8: Illustration of an exercise date that falls between two payment dates

In reality though, the full cash flow between dates t_1 and t_2 is lost, so the swaption has to be priced with this cash flow included, since the swaption models all cash flows that have to be replaced after the counterparty's default. Stein and Lee (2010:13-14) proposed a solution to this by comparing a forward start swap and a tail swap. If the forward start swap is entered into between two payment dates, as illustrated above, the payment will be prorated as explained. In a tail swap, the whole cash flow between t_1 and t_2 will take place. The formulae for the time t values of the forward start swap, $S(t)$, is given as

$$\boldsymbol{S(t,t,T) = \sum_{i=1}^{n-1} f(t,t_i,t_{i+1}) D(t,t,t_{i+1}) h_i - C \sum_{i=1}^{n-1} D(t,t,t_{i+1}) h_i} \tag{8}$$

Where

- $t_1 = t$
- C is the swap rate
- h_i is the length of period i, expressed in years

Alternatively, it can be written as

$$\boldsymbol{f(t,t,t_2)D(t,t,t_2)h_1} + \sum_{i=2}^{n-1} f(t,t_i,t_{i+1})D(t,t,t_{i+1})h_i - \boldsymbol{C}.(\boldsymbol{D(t,t,t_2)h_1} + \sum_{i=2}^{n-1} D(t,t,t_{i+1})h_i)$$

$\boldsymbol{\overline{S}(t,t,T)}$ can also be expressed with formula (8), except that in this case, $\boldsymbol{t_1}$ is the last reset date before date t, as shown in Figure 2-8. $\boldsymbol{\overline{S}(t,t,T)}$ can therefore be written as

$$\boldsymbol{f(t,t_1,t_2)D(t,t_1,t_2)h_1} + \sum_{i=2}^{n-1} f(t,t_i,t_{i+1})D(t,t,t_{i+1})h_i - \boldsymbol{C}.(\boldsymbol{D(t,t_1,t_2)h_1} + \sum_{i=2}^{n-1} D(t,t,t_{i+1})h_i)$$

In order to transform $S(\mathrm{t},t,T)$ to $\bar{S}(\mathrm{t},t,T)$, it is convenient to find the difference between the two swaps and add it to $S(t,t,T)$. Stein and Lee (2010:14) noted however that the difference can be approximated by the time 0 value of the difference between the two swaps. Using the alternate definition of $S(t,t,T)$ and that of $\bar{S}(t,t,T)$, this can be expressed as

$$\boldsymbol{D_{tail}(t) = \overline{S}(0,t,T) - S(0,t,T) = L(\frac{D(0,t,t_2)}{D(0,t_1,t_2)} - 1 - C.(t-t_1)D(0,t,t_2))} \quad \textbf{(9)}$$

Since it will be assumed that all fixed and floating rate payments occur at the same times, there will be no distinction between floating and fixed rate periods from here on, in the case of a_1 and $\overline{a_1}$. Since the price of a swaption is the expected payoff at time t, the swaption price at time 0 can be given as

$$\boldsymbol{P(0,t,T) = \xi(0)E_0\left[\frac{\max(S(t,t,T),0)}{\xi(t)}\right]} \quad \textbf{(10)}$$

where $\xi(t)$ is the value of the chosen numeraire at time t.

Now, taking into account the difference between the forward start swap and a tail swap entered into at time t, given by $D_{tail}(t)$, a substitution must be made into the formula above. This is because the swaption used in the CVA calculation will have a tail swap as underlying asset, instead of a forward start swap as an underlying asset.

Since

$D_{tail}(t) = \bar{S}(0,t,T) - S(0,t,T)$, we obtain $\bar{S}(0,t,T) = D_{tail}(t) + S(t,T)$, therefore

$$\boldsymbol{\max(\overline{S}(t,t,T),0) = \max(D_{tail}(t) + S(t,t,T),0) = \max(S(t,t,T),-D_{tail}(t)) + D_{tail}(t)} \quad \textbf{(11)}$$

Therefore, to price a swaption with the tail swap as underlying, equation (11) must be substituted into equation (10). This finally leads to

$$\boldsymbol{P_{tail} = \xi(0)E_0\frac{\max(\overline{S}(t),0)}{\xi(t)} = \xi(0)E_0\left[\frac{\max(D_{tail} + S(t),0)}{\xi(t)}\right]} \quad \textbf{(12)}$$

Equation (12) will be expanded in section 2.3, where the pricing of swaptions under the Hull-White one-factor model is explained in detail.

2.2.3 Major factors affecting CVA

Since CVA depends on the price of swaptions, the usual factors that affect the price of a swaption, will affect CVA, such as volatility of interest rates (or swap rates), time to expiry of the swaption (how far into the future the possible default date is), swap rate of the underlying swap (current swap rates) and the strike price (swap rate of the swap contract that is being analysed). Refer to sections 2.3 and 2.4 for more details on these factors in the context of pricing EE. Probability of default will also play a major role in the magnitude of the CVA (Sorensen and Bollier, 1994:5). Most of these factors will be examined in the results section (Section 5).

Sorensen and Bollier (1994:5) also named the shape of the term structure as a major determinant of the shape of the expected exposure profile, and thus CVA. If the term structure has a rising shape, as shown in Figure 2-10, it implies that interest rates are expected to rise for at least the medium term future. Swap rates are theoretically calculated by finding a fixed payment value such that the present value of all expected future fixed and floating rate payments in the swap are equal. The swap rate will therefore be higher than the expected floating rate for a certain time period at the beginning of the swap, as shown in Figure 2-9 below, left.

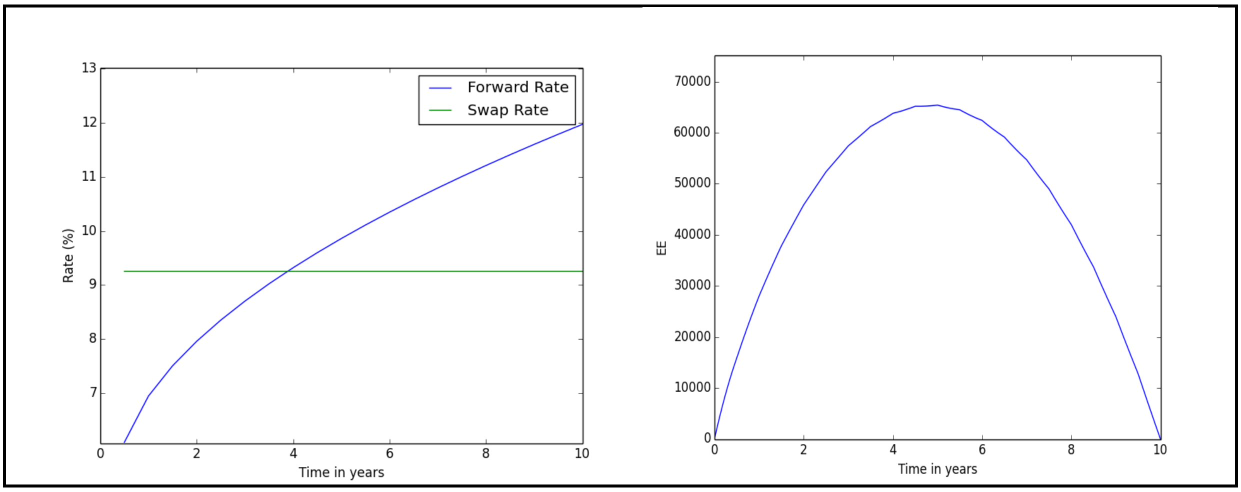


Figure 2-9: Left: Swap and forward rate applicable to each payment date in the swap contract. Right: NPV at each respective time.

The specific length of this time period will be determined by the specific shape of the zero curve. The party that is long the swap, e.g. party A will be expected to make a loss on the payments during this time period, since payments based on the swap rate will be lower than receivables, which are based on the floating rate. However, it is expected to be "compensated" for this loss in the latter time period of the swap, when floating rates are expected to be higher than the fixed rate. At the end of the first time period in the swap, though, the swap will be expected to have the greatest net present value to party A that it will ever have during its lifetime, because A expects to make profits on each netted cash flow from that time until the end of the swap. Party A's exposure will therefore be expected to be greatest at this point (or near it, since discount factors also affect the value of the NPV). This is shown in Figure 2-9 above, right. Note that this "analysis" is done assuming that today's expectations of future short rate movements will be met, which implies that volatility is zero.

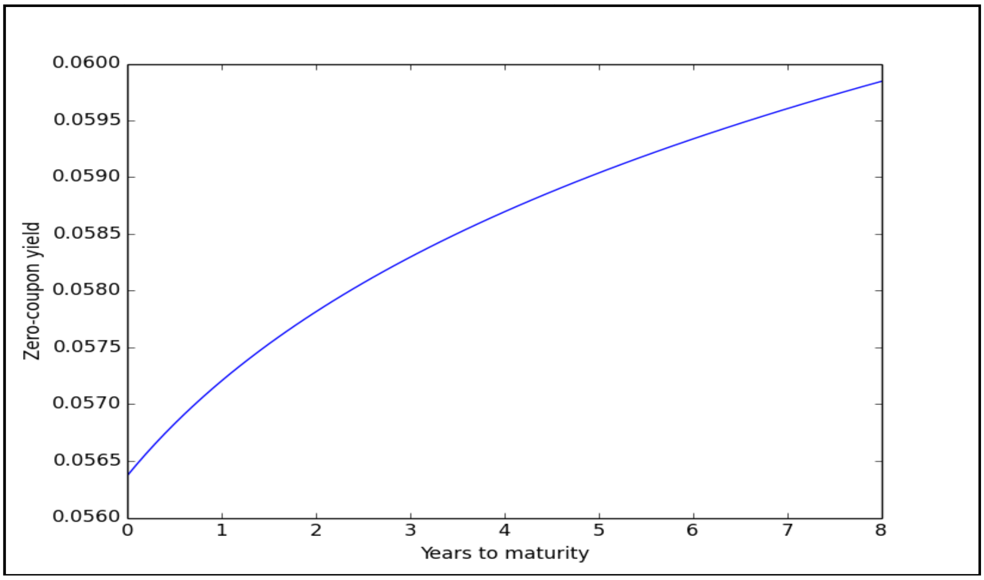


Figure 2-10: Rising term structure (expectation that rates will increase)

The opposite of the scenario just discussed will be applicable if the term structure exhibited a downward slope, as shown below,

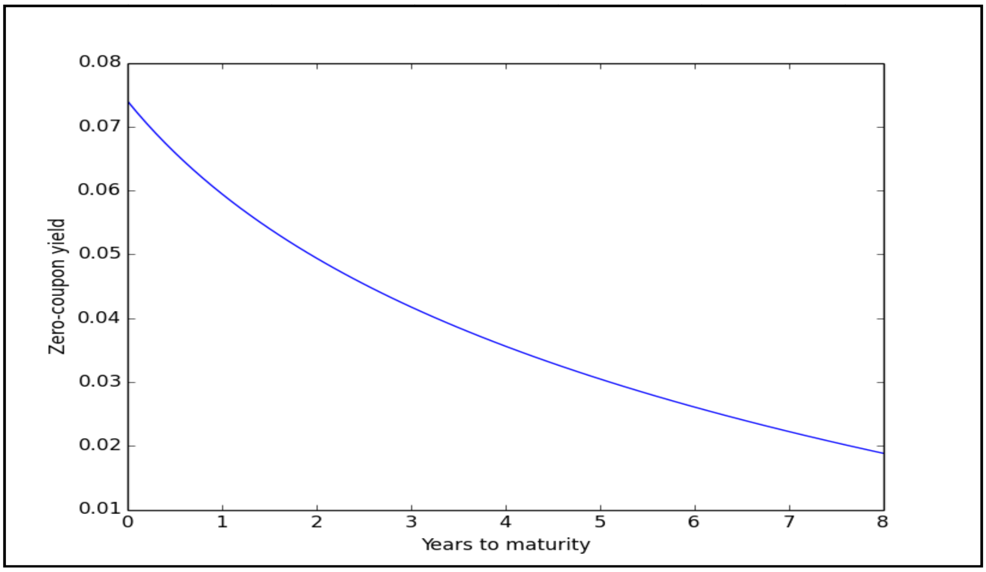


Figure 2-11: Falling term structure (expectation that rates will decrease)

since now the receiver of fixed payments (who is short the swap) will carry the exposure in a similar manner as the payer of fixed did in the previous scenario.

In a flat term structure, the expected exposure, assuming the term structure will not change, is zero at every point in time during the swap.

The fact that the term structure has such a significant effect on CVA is one of the main reasons Sorensen and Bollier (2014:5) stated that CVA should be analysed under a term structure model, which has been defined in section 1.3.

The Hull-White model has the ability to characterise the likely behaviour of the short rate with volatility and mean reversion parameters, as well as being able to mimic the observed term structure. The Hull-White model thus allows the term structure to change in the future, which is a realistic scenario, since expectations change all the time. Approximating EE by just using NPV, on the other hand, does not take changing expectations into consideration. The volatility and mean reversion parameters are to be estimated by matching swaption prices implied by the model to swaption prices observed in the market. This is known as calibrating the model, and will be discussed at a later stage in the literature review.

2.3 The Hull-White one-factor model

The Hull-White one-factor model is a term structure model, specifically a no-arbitrage model, which describes possible evolutions of the short rate over time. Hull (2012:691-692) defines the process for the short rate in the Hull-White one factor model as

$$\boldsymbol{dr = \left(\theta(t) - ar(t)\right)dt + \sigma dz} \tag{13}$$

The equation above can be adapted to describe changes in the short rate over measurable changes in time (e.g. daily changes in short rates), which is known as the discretised version of the equation

$$\boldsymbol{\Delta r = \left(\theta(t) - ar(t)\right)\Delta t + \sigma\sqrt{\Delta t}\, Z} \tag{14}$$

where

- $\theta(t) = F_t(0,t) + aF(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at})$
  - $F(0,t)$ is the instantaneous forward rate at time t, as seen at time 0
  - $F_t(0,t)$ is the first partial derivative of $F(0,t)$ with respect to t.
- a is the mean reversion rate
- $r(t)$ is the short rate at time t
- Δt is the incremental change in time, measured in years
- σ is the volatility parameter
- $Z \sim N(0,1)$

2.3.1 European swaption pricing under the one-factor model

Consider a european swaption that matures at time T, where the underlying swap matures at time T_M. Given model parameters a and σ, the price of a swaption, denoted $\boldsymbol{P_{HW}\,(0, T, T_M)}$, is given by Bateson (2011:311) as

$$L\left(D(0,0,T)N(-Z^*) - D(0,0,T_M)N(-Z^* - v_m) - K\sum_{t_i>T}^{T_M} D(0,0,t_i)h_i N(-Z^* - v_i)\right) \quad \textbf{(15)}$$

while the price of a european receiver swaption, denoted $\boldsymbol{R_{HW}(0, T, T_M)}$, holding all other factors constant, is given by

$$L\left(-D(0,0,T)N(Z^*) + D(0,0,T_M)N(Z^* + v_m) + K\sum_{t_i>T}^{T_M} D(0,0,t_i)h_i N(Z^* + v_i)\right) \quad \textbf{(16)}$$

where

- L is the notional value on the swap
- $N(Z)$ is the cumulative standard normal distribution function
- t_i is the time (in years) of the i-th swap payment
- Z^* is the value of Z which satisfies the boundary condition of the swaption's exercise
- v_i is volatility of the discount bond price at time T, maturing at time t_i

Boundary condition, above, refers to the condition where the swaption is at-the-money at the exercise date, as seen from the valuation date.

The forward volatility of the discount bond price, v_i, is derived from the following expression, given by Bateson (2011:306) as

$$v^2(t,s,T) = \sigma^2\left(\frac{1-\exp\left(-a(T-s)\right)}{a}\right)^2\left(\frac{1-\exp\left(-2a(s-t)\right)}{2a}\right) \quad \textbf{(17)}$$

where $v(t, s, T)$ is interpreted as the volatility of the discount bond price at time s, which matures at time T, as seen at time t.

All the values in the swaption price formula (15) should be readily available, except Z^*. It will now be discussed how this value could be obtained.

Z^* is the boundary condition for exercise, which is required to integrate (thus, find the expected value of) the swaption's value at expiry. The expression below, which will be derived fully in Appendix A, determines the swaption's price at expiry

$$\boldsymbol{Max((1 - D(0,T,T_M),0) - K\sum_{t_i>T}^{T_M} D(0,T,t_i)\,h_i,0)} \quad \textbf{(18)}$$

The point where the expression above changes from zero to a positive value, is the boundary point of exercise. This point is given by

$$\boldsymbol{1 - D(0,T,T_M) = K\sum_{t_i>T}^{T_M} D(0,T,t_i)\,h_i} \quad \textbf{(19)}$$

or in terms of Z^*,

$$\boldsymbol{1 - \lambda(T,T_M,Z^*) = K\sum_{t_i>T}^{T_M} \lambda(T,t_i,Z^*)\,h_i} \quad \textbf{(20)}$$

Since $D(0,t,T)$ is stochastic if $t \neq 0$, it would thus be the only determinant of the above expression's value after the given values have been established. It is given by

$$\boldsymbol{D(0,t,T) = \frac{D(0,0,T)}{D(0,0,t)}\exp\left(-\frac{1}{2}v^2(0,t,T) - v(0,t,T)Z\right)} \quad \textbf{(21)}$$

It makes sense to re-parameterise the expression above with Z as a parameter, therefore we have

$$\boldsymbol{\lambda(T,T_M,Z^*) = \frac{D(0,0,T_M)}{D(0,0,T)}\exp\left(-\frac{1}{2}v^2(0,T,T_M) - v(0,T,T_M)Z^*\right)}$$

Note that Z is the only unknown, or stochastic variable in expression (21), so therefore it follows that Z is the only variable element in expression (19), so therefore a value Z^* must be found such that (20) holds. This value Z^* can be found by a root finding algorithm, which was implemented with a package in the Python language. Source text for the program used to find the price of a swaption under the Hull-White model can be found in Appendix C.

2.3.2 Adjusting for accrued and partial payments lost (Tail Swaption)

Consider a payer swap, currently held, that matures at time T_M. Assume that accrued payments are lost, and a default at time T occurs between two payment dates, T_0 and T_1. The EE at time T, denoted $\boldsymbol{P_{HW\,tail}(0,T,T_M)}$, is given by

$$\begin{aligned} &\boldsymbol{D(0,0,T_0)N(-Z^{**}+\bar{v}-v)\exp\left(v\bar{v}-\bar{v}^2\right) - D(0,0,T)N(-Z^{**})} \\ &\boldsymbol{- K(T-T_0)D(0,0,T_1)N(-Z^{**}-v) + P^*_{HW}(0,T,T_M)} \end{aligned} \tag{22}$$

In the case of a receiver swap, keeping all other factors constant, the EE at time T, denoted $\boldsymbol{R_{HW\,tail}(0,T,T_M)}$, is given by

$$\begin{aligned} &\boldsymbol{-D(0,0,T_0)N(Z^{**}-\bar{v}+v)\exp\left(v\bar{v}-\bar{v}^2\right) + D(0,0,T)N(Z^{**})} \\ &\boldsymbol{+ K(T-T_0)D(0,0,T_1)N(Z^{**}+v) + R^*_{HW}(0,T,T_M)} \end{aligned} \tag{23}$$

where $v = v(0,T,T_1)$ and $\bar{v} = v(0,T_0,T_1)$, and Z^{**} is the value of Z that satisfies

$$\boldsymbol{\frac{\lambda(0,T,Z^*)}{\lambda(0,T_0,Z^*)} - 1 - K(T-T_0)\lambda(T,T_1,Z^*) + 1 - \lambda(T,T_M,Z^*) = K\sum_{t_i>T}^{T_M}\lambda(T,t_i,Z^*)\,h_i} \tag{24}$$

$\boldsymbol{P^*_{swaption}(0,T,T_M)}$ and $\boldsymbol{R^*_{swaption}(0,T,T_M)}$ are equations (15) and (16), respectively, except that instead of Z^*, we will use Z^{**}. Note that if default occurs on a payment date, the tail swaption becomes an ordinary swaption, described in the previous subsection.

Equations (22) and (23) were derived using Stein and Lee's (2010:14) suggestion that EE can be calculated by altering the expectation to calculate a swaption price to

$$\boldsymbol{P(0,T,T_M)_{HW} = \xi(0)\mathrm{E}_0\frac{\max(\bar{\mathrm{S}}(\mathrm{T},T,T_M)\,,0)}{\xi(T)} = \xi(0)E_0\left[\frac{\max(D_{tail}+S(0,T,T_M),0)}{\xi(T)}\right]} \tag{12}$$

where $D_{tail}(T)$ (from section 2.2.2) is the difference between $S(0,T,T_M)$ and $\bar{S}(0,T,T_M)$, and is equal to

$$\boldsymbol{\bar{S}(0,T,T_M) - S(0,T,T_M) = L\left(\frac{D(0,T,T_1)}{D(0,T_0,T_1)} - 1 - K(T-T_0)D(0,T,T_1)\right)} \tag{9}$$

Note that slightly different (simplified) swaps were considered compared to the ones that Stein and Lee used in their paper. Stein and Lee used swaps with different payment periodicities and frequencies, while in this paper, the simplification is made that fixed and floating payments will occur on the same dates, hence the slight difference in the formulation for the swap formulae. A full derivation for equations (22) and (23) are shown in Appendix A.

2.3.3 Adjusting for collateral agreements

Consider a payer swap, currently held and maturing at time T_M, with regular collateral posting. Let the collateral threshold amount be defined by H. Also, for simplicity, disregard the grace period and assume that collateral is posted on every date that EE is calculated. Assuming partial payments cannot be recovered (see section 2.2.2) exposure on date T, denoted $\boldsymbol{R_{HW\,tail}(0,T,T_M)}$, is then given by

$$\boldsymbol{P_{HW\,tail}(0,T,T_M) - P^{**}{}_{HW\,tail}(0,T,T_M) + HN(-Z^{***})} \quad \textbf{(25)}$$

And for a receiver swap, denoted $\boldsymbol{R_{collateral}(0,T,T_M)}$, keeping all other factors constant,

$$\boldsymbol{R_{HW\,tail}(0,T,T_M) - R^{**}{}_{HW\,tail}(0,T,T_M) + HN(Z^{***})} \quad \textbf{(26)}$$

Where Z^{***} is the value for Z that satisfies

$$\boldsymbol{\frac{\lambda(T,T_1,Z^{**})}{\lambda(T_0,T_1,Z^{**})} - 1 - K(\mathrm{T}-\mathrm{T}_1)\lambda(T,T_1,Z^{**}) + 1 - \lambda(T,T_M,Z^{**}) = K\sum_{t_i>T}^{T_M}\lambda(T,t_i,Z^{**})\,h_i + \frac{H}{L}}$$

with H the threshold amount. $\boldsymbol{P^{**}{}_{HW\,tail}(0,T,T_M)}$ and $\boldsymbol{R^{**}{}_{HW\,tail}(0,T,T_M)}$ are equations (22) and (23) respectively, except that Z^{***}, instead of Z^{**} will be used.

The formula above was derived from Gibson's (2005:6) suggestion that under collateral agreements, EE can be calculated by taking expectation of

$$\boldsymbol{\xi(0)E\left[\frac{\max\left(\bar{S}(T,T,T_M) - \max[\bar{S}(T,T,T_M) - H, 0], 0\right)}{\xi(T)}\right]}$$

A full derivation is given in Appendix A.

2.3.4 Expected exposure DV01

Henrard (2005:2) derived an approximation for the delta of a payer swaption as

$$-\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{r_j}} = \sum_{i=0}^{n} \boldsymbol{c_i N(-Z^* - v_i)} \frac{\partial \boldsymbol{D(0, t_i)}}{\partial \boldsymbol{r_j}} \quad \textbf{(27)}$$

where,

- $c_i = h_i K$ for $1 \leq t < n$, $c_0 = -1$, and $c_n = 1 + h_i K$
- r_j is the j-th factor that affects the term structure

Similarly, the approximation for the delta of a receiver swaption is given as

$$\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{r_j}} = \sum_{i=0}^{n} \boldsymbol{c_i N(Z^* + v_i)} \frac{\partial \boldsymbol{D(0, t_i)}}{\partial \boldsymbol{r_j}} \quad \textbf{(28)}$$

where the parameters have the same meaning as with the swaption pricing formulae.

The r_j parameter needs some further clarification. It can be any factor that affects the term structure, for example forward rates or zero-coupon bonds. In the case of forward rates, for example, the delta of the swaption with respect to the forward rate that prevails between 15 and 18 months from now, might be sought, so then r_j would represent this forward rate. In particular, in this paper r_j will represent a parallel shift in the yield curve, which is an allowable choice for r_j, according to Henrard (2005:2).

In the context of the calculation of CVA, the sum of the delta of all the swaptions that make up the portfolio for the purposes of CVA calculation can be seen as the change in CVA resulting from an incremental change in r_j.

In this paper, the delta formula will be used to calculate a DV01 measure instead, which is the change in a swaption's value for a basis point drop at every point on the zero curve, using forward difference for the $\frac{\partial D(0,t_i)}{\partial r_j}$ term. This as a whole will be described further in the research methodology, but forward difference will be defined in the next subsection. Note that from now on, a parallel shift in the yield curve by n units will be represented by $w(n)$. More specifically, a basis point shift will be $w(bp)$. We will therefore write equation (27) alternatively as

$$-\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{w(bp)}} = \sum_{i=0}^{n} \boldsymbol{c_i N(-Z^* - v_i)} \frac{\partial \boldsymbol{D(0, t_i)}}{\partial \boldsymbol{w(bp)}} \quad \textbf{(27)}$$

with the same modification done to equation (28), if the alternative form is preferred.

2.3.5 Forward difference

A derivative can be estimated numerically by difference methods. Specifically, the forward difference method will be of interest in this paper. Urroz (2001:32) defines forward difference as

$$\frac{dg}{dx} = \frac{g(x_0 + h) - g(x_0)}{h}$$

where

- g is a continuous function at x_0
- h is a very small number (tending to infinity, ideally)

For calculating DV01, though, we will use the formula above, since we want to find the amount by which $f(x_0)$ changes at the point x_0 for each 0.0001 (basis point) decrease in x_0. The following formula will therefore be used

$$\boldsymbol{\frac{df}{dx}(0.0001) = DV01 = \frac{g(x_0 + h) - g(x_0)}{h}(0.0001)} \quad \textbf{(29)}$$

2.3.6 Netting

Only netting cases as described in section 2.1.2.1 were considered for the Hull-White model. The principal values for each swap that the party holds at the time that EE is calculated for, were simply added. This new value was then used as the principal for one swap contract, and EE was calculated accordingly.

2.3.7 Model Calibration

In order to make the model relevant for the situation that an institution would find itself in, the model has to be calibrated to available swaption prices in the market that that the institution operates in. This means that the mean reversion parameter, a, and the volatility parameter, σ, must be found such that swaption prices implied by the model resembles swaption prices in the market as closely as possible. In a paper by Gurrieri, Nakabayashi and Wong (2009:12-14), three methods for the calibration of the Hull-White one-factor model to swaption prices are proposed.

- Method 1

Using this method, a value for σ is estimated while mean reversion (a) is held fixed at an arbitrary constant specified by the analyst. The following expression is minimized

$$\boldsymbol{G_a(\sigma) = \sum_{i=1}^{n_m}\sum_{j=1}^{n_{t-1}} W_{i,j}\left(\frac{PV_a^{mod}(M_i, Y_j)(\sigma)}{PV^{mkt}(M_i, Y_j)} - 1\right)^2} \quad \textbf{(30)}$$

where

- M_i is the i-th maturity considered
- Y_j is the j-th tenor considered
- $PV^{mkt}(M_i, T_j)$ is the observed market price of a swaption with maturity M_i and tenor T_j
- $PV_a^{mod}(M_{i,}T_j)(\sigma)$ is the price of a swaption under the Hull-White one-factor model with maturity M_i and tenor T_j, and fixed reversion rate a and volatility σ
- $W_{i,j}$ is the weight assigned to a swaption with maturity M_i and tenor T_j

- Method 2

This method relies on first finding reversion rate a by an approximation method, then finding σ such that swaption prices generated by the model resembles market swaption prices as closely as possible. The approximation method relies on the following result

$$\frac{V_{swap}(M_i, T_j)}{V_{swap}(M_i, T_k)} = \left[\frac{(D(0, M_i) - D(0, Y_k))B((M_i, Y_j)}{\left(D(0, M_i) - D(0, Y_j)\right)B(M_i, Y_k)}\right]^2 \quad \textbf{(31)}$$

where

- $V_{swap}(M_i, T_j)$ is the approximated variance of a swap, based on implied volatility from swaptions with maturity M_i and tenor T_j, using a model with a specific mean reversion
- $B(M_i, T_j) = \frac{1-\exp(a(t-T))}{a}$

The value of this equation is independent of the volatility, which makes its usage appropriate for finding the volatility in a step-by-step procedure, as is the case here. Equation (31) above gives the ratio between the variances of two specified swaps. Thus the square root of the equation above is taken to find the volatility ratio, which is then compared with market implied volatilities, using the equation below, which will then be minimized to find the appropriate value for the mean reversion parameter.

$$F(a) = \sum_{i=1}^{n_m} \sum_{j=1}^{n_t - 1} \frac{W_{i,j+1}}{W_{i,j}} \left(\sqrt{\frac{V_{swap}(M_i, Y_{j+1})}{V_{swap}(M_i, Y_j)}(a)} - \frac{IV_{i,j+1}}{IV_{i,j}} \right)^2 \quad \textbf{(32)}$$

where,

- $IV_{i,j}$ is the market implied volatility for a swaption with maturity M_i and tenor T_j

Hence the value for a that gives the lowest value for $F(a)$ is the calibrated mean reversion parameter.

The next step is to find the volatility parameter. This is done by using method 1, but using the mean reversion a found in step 1 of method 2, instead of an arbitrary value for a.

- Method 3

Method 3 involves optimizing for the mean reversion and volatility at the same time by minimizing the following formula with a and σ as variable parameters

$$\boldsymbol{G(\sigma, a) = \sum_{i=1}^{n_m} \sum_{j=1}^{n_{t-1}} W_{i,j} \left(\frac{PV_a^{mod}(M_i, Y_j)(\sigma, a)}{PV^{mkt}(M_i, Y_j)} - 1 \right)^2} \quad \textbf{(33)}$$

To increase efficiency, method 2 could first be employed to find initial guesses for a and σ.

Gurrieri, Nakabayashi and Wong found that method 2 adapted very well to changes in behaviour of interest rates. While method 3 generally gave the highest accuracy regarding swaption prices generated by the model, its computation took much longer than that of method 2. Method 1 performed similar to method 3 as long as the swap market's behaviour remained relatively unchanged for the period from which swaption data used for calibration, was extracted.

Note that Gurrieri, Nakabayashi and Wong also found that the three calibration methods used can also be used for time dependent sigma and mean reversion parameters. In this paper, however, only constant a and σ will be used.

2.4 Black's model

When using Black's model to price interest rate derivatives, two key assumptions are made. The first is that forward rates are lognormally distributed. The second, and most important, is that forward starting swap rates are lognormally distributed. The assumption that forward rates are log-normally distributed lies on the following assumed stochastic formula, given by Bateson (2011:303) for percentage change in forward rates

$$\boldsymbol{\frac{df(t,T,T+h)}{f(t,T,T+h)} = -\sigma(T,T+h)dz(t)} \quad \textbf{(34)}$$

It can be shown from the formula above that forward rates are given by the following

$$\boldsymbol{f(t,T,T+h) = f(0,T,T+h)\exp\left(-\frac{v^2}{2} - vZ\right)} \quad \textbf{(35)}$$

where,

- $v = \sigma(T,T+h)\sqrt{t} = \sigma\sqrt{t}$

- $\sigma(T, T+h)$ is the volatility of the percentage change in forward rates between times T and $T+h$. It will be assumed constant.
- Z is a standard normal random variable

2.4.1 European swaption pricing under Black's model

Hull (2012:660-661) gives the current value of a european payer swaption, denoted $\boldsymbol{P_{BS}(0, T, T_M)}$, maturing at time T, with underlying swap maturing at time T_M, given volatility parameter σ under Black's model as

$$\boldsymbol{LA(0)[s_0 N(d_1) - s_K N(d_2)]} \quad \textbf{(36)}$$

The value of a european receiver swaption, denoted $\boldsymbol{R_{BS}(0, t, T)}$, given the same underlying and model parameters, is given as

$$\boldsymbol{LA(0)[KN(-d_2) - s_0 N(-d_1)]} \quad \textbf{(37)}$$

where,

- $A(t)$ is defined as

$$\boldsymbol{A(t) = \sum_{i=0}^{M-1} (T_{i+1} - T_i) D(t, T_{i+1})} \quad \textbf{(38)}$$

- s_0 is defined as

$$\boldsymbol{s_0 = \frac{D(0, T_0) - D(0, T_N)}{A(t)}} \quad \textbf{(39)}$$

- d_1 is defined as

$$\boldsymbol{d_1 = \frac{\ln\left(\frac{s_0}{s_K}\right) + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}} \quad \textbf{(40)}$$

- And d_2

$$\boldsymbol{d_2 = \frac{\ln\left(\frac{s_0}{s_K}\right) - \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}} \quad \textbf{(41)}$$

2.4.2 Adjusting for accrued and partial payments lost (Tail Swap)

Consider a payer swap, currently held, that matures at time T_M. Assume that accrued and partial payments are lost, and that a default at time T occur between two payment dates, T_0 and T_1. The EE at time T, denoted $\boldsymbol{P_{BS\,tail}(0,T,T_M)}$, is given by

$$\boldsymbol{A(0)L\left(\left(s_0+\frac{(T-T_0)f_0}{A(0)}\right)N(d_1)-\left(K+\frac{(T-T_0)K\,D(0,T,T_1)}{A(0)}\right)N(d_2)\right)} \quad \textbf{(42)}$$

For a receiver swap, keeping all other factors the same, the EE, denoted $\boldsymbol{R_{BS\,tail}(0,T,T_M)}$, is given by

$$\boldsymbol{A(0)L\left(\left(K+\frac{(T-T_0)\mathrm{K\,D}(0,\mathrm{T},\mathrm{T}_1)}{A(T)}\right)N(-d_1)-\left(s_0+\frac{(T-T_0)f_0}{A(T)}\right)N(-d_2)\right)} \quad \textbf{(43)}$$

where f_0 is equivalent to $f(0,T_0,T)$, the annual forward interest rate between T_0 and T, as seen at time 0, and

$$\boldsymbol{d_1=\frac{\ln\left(\frac{\left(s_0-\frac{f_0}{A(0)}\right)}{\left(K+\frac{K(T-T_1)D(0,T,T_1)}{A(0)}\right)}\right)+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}}$$

$$\boldsymbol{d_2=\frac{\ln\left(\frac{\left(s_0-\frac{f_0}{A(0)}\right)}{\left(K+\frac{K(T-T_1)D(0,T,T_1)}{A(0)}\right)}\right)-\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}=d_1-\sigma\sqrt{T}}$$

As in the Hull-White case, we used $D_{tail}(T)$ to transform $S(0,T,T_M)$ to $\bar{S}(0,T,T_M)$, then took the expectation to find the value for a tail swaption, as given in Stein and Lee (2010:14), to represent EE:

$$\boldsymbol{\xi(0)E_0\left[\frac{\max(S(T,T,T_M)+\ D_{tail}(T),0)}{\xi(T)}\right]=\ A(0)L\ E\left[max\left(s_t-\ K+\frac{D_{tail}(T)}{A(T)},0\right)\right]}$$

A full proof for this formula, and for the ordinary swaption formula can be found in Appendix A.

2.4.3 Adjusting for collateral agreements

Consider a payer swap, currently held and maturing on date T_M, with regular collateral posting. Let the collateral threshold amount be defined by H. Also, for simplicity, disregard the grace period and assume that collateral is posted on every date that EE is calculated. Assuming that accrued payments and partial payments cannot be recovered, the EE on date T, denoted $\boldsymbol{P_{BS\,collateral}(0,T,T_M)}$, is given as

$$\boldsymbol{P_{tail}(0,T,T_M) - A(0)L\left(\left(s_0 + \frac{(T-T_0)f_0}{A(T)}\right)N(d_1) - \left(\frac{H}{LA(T)} + K + \frac{K(T-T_0)D(0,T,T_1)}{A(T)}\right)N(d_2)\right)} \quad \textbf{(44)}$$

In the case of a receiver swap, keeping all other factors constant, EE on date T, denoted $\boldsymbol{R_{BS\,collateral}(0,T,T_M)}$, is given by

$$\boldsymbol{R_{tail}(0,T,T_M) - A(0)L\left(\left(\frac{H}{LA(T)} + K + \frac{K(T-T_0)D(0,T,T_1)}{A(T)}\right)N(-d_2) - \left(s_0 + \frac{(T-T_0)f_0}{A(T)}\right)N(-d_1)\right)} \quad \textbf{(45)}$$

In both cases,

$$d_1 = \frac{\boldsymbol{\ln\left(\frac{\left(s_0 - \frac{f_0}{A(0)}\right)}{\left(\frac{H}{LA(0)} + K + \frac{K(T-T_0)D(0,T,T_1)}{A(0)}\right)}\right) + \frac{\sigma^2}{2}T}}{\boldsymbol{\sigma\sqrt{T}}}$$

$$d_2 = \frac{\boldsymbol{\ln\left(\frac{\left(s_0 - \frac{f_0}{A(0)}\right)}{\left(\frac{H}{LA(0)} + K + \frac{K(T-T_0)D(0,T,T_1)}{A(0)}\right)}\right) - \frac{\sigma^2}{2}T}}{\boldsymbol{\sigma\sqrt{T}}}$$

Equations (44) and (45) were derived using Gibson's (2005:6) suggestion that EFE can be calculated by taking expectation of

$$\boldsymbol{\xi(0)E\left[\frac{\max\left(\bar{S}(T,T,T_M) - \max[\bar{S}(T,T,T_M) - H, 0], 0\right)}{\xi(T)}\right]}$$

The full derivation is given in Appendix A.

2.4.4 Netting

Netting under Black's model was handled exactly the same as described in section 2.3.6 under the Hull-White model.

2.4.5 Calculation of expected exposure DV01

A numerical approximation was used for the DV01 measure for CVA. No solid literature could be found on a practical calculation method, and was derived primarily from the definition of DV01, which is defined as the dollar value of a basis point change in interest rates (Hull, 2012:804). To clarify, it is the amount by which an instrument changes when a parallel basis point shift occurs in the yield curve. Since the method is an improvisation, it is outlined in the research methodology under section 4.5.2.

2.4.6 Model calibration

Black's model was calibrated by directly using implied volatilities relevant to the swaption's maturity and tenor. The actual implied volatility matrices used on each date are listed in Appendix B. This method was used because according to Hull(2012:660-661), implied volatilities are obtained by using the actual swaption price in the market as an input into Black's model, then finding the volatility that would have produced that swaption price as an output. The exact interpolation method is outlined under section 4.3.2 in the research methodology, since the interpolation method was also an improvisation.

Note that when calculating CVA on a swap, this method necessarily implies that the parameters for Black's model will be time-dependent, since the tenor of the swap changes as time passes on.

2.5 Summary

A relatively comprehensive summary of the knowledge necessary to understand the intricacies of CVA calculation was given in this literature review. The basic theory of simplistic IRS pricing was handled first, along with how risks involved in holding swap contracts can be mitigated. This was followed by an overview of the swaption approach to calculating CVA. A more detailed treatment of CVA calculation followed. This was done so that the swaption approach as a whole would make better sense to the reader, after which the mathematics that follows would be easier to understand.

The detailed treatment of CVA calculation involved adjustments made to the underlying swaps in the swaption contracts, a discussion of factors that affect the value of CVA, and then a detailed description of the Hull-White and Black models used in the swaption valuation. This description included the basic formulae for pricing swaptions, as well as the exact adjustments to each formula to provide for collateral and netting agreements. Each of these sections then concluded with discussions of and formulae for EE DV01 calculation and model calibration. In certain cases, where no solid academic literature could be found on exact calculation methods, improvised methods were used. These will be discussed in the research methodology in section 4.

3 Rationale behind using swaption approach

Calculating a value for CVA (using the semi-analytical method) is clearly much more effort than just using current market values for the current swap portfolio. The justification for using the more involved semi-analytical method is outlined in this chapter.

The textbook method for calculating current swap values is to use equation (5)

$$\boldsymbol{C(0,0,n)} = \frac{\mathbf{1 - D(0,0,}\boldsymbol{n}\mathbf{)}}{\sum_{i=1}^{mn} \boldsymbol{D}(\mathbf{0,0,}\frac{i}{\mathrm{m}})\boldsymbol{h}}$$

The equation above is derived from the assumption that at inception, the expected floating portion and fixed portion of a swap contract have to be equal. The expected floating rates implied from the term structure (which is a reflection of current market expectations of future interest rates) are used for the floating rate payments, in the simplest implementation of swap rate calculation. This does not take into account that expectations may change, which can be thought of as the "volatility" of expectations.

"Volatility" of expectations can be gauged from the volatility parameter σ of a calibrated term structure model, such as the Hull-White model used in this paper. In the case of the Hull-White model, σ is used to describe the volatility of the short rate, which will directly influence the volatility of the term structure. The term structure's volatility will influence the volatility of the swap rate, which will in turn influence the volatility of exposure to counterparty default risk. To illustrate, consider Figure 3-2. It is a simulation of 20 different structures that could be observed one year from the present date, ending ten years from the present date. Similarly, Figure 3-3 shows simulated term structures that could be observed 2 years from now. These term structures were each used to calculate an implied swap rate for each simulation one and two years from now, respectively. These are shown in Figure 3-4. The Hull-White one-factor model, with

- σ = 1%
- a = 5%
- Increments between payment dates as 6 months
- And a commonly observed, but fictional term structure (Figure 3-2)

was used to simulate these different term structures using equation (21), which allows future zero coupon bond prices to be stochastic.

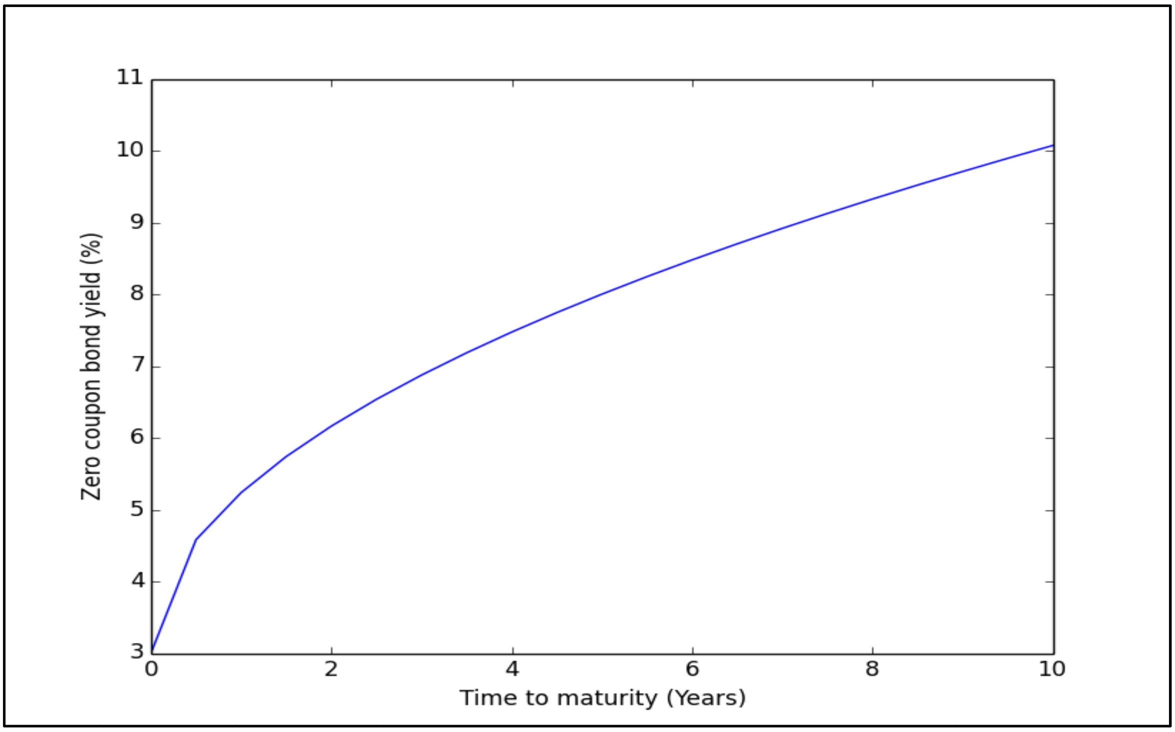


Figure 3-1: Rising term structure used for simulations in this chapter

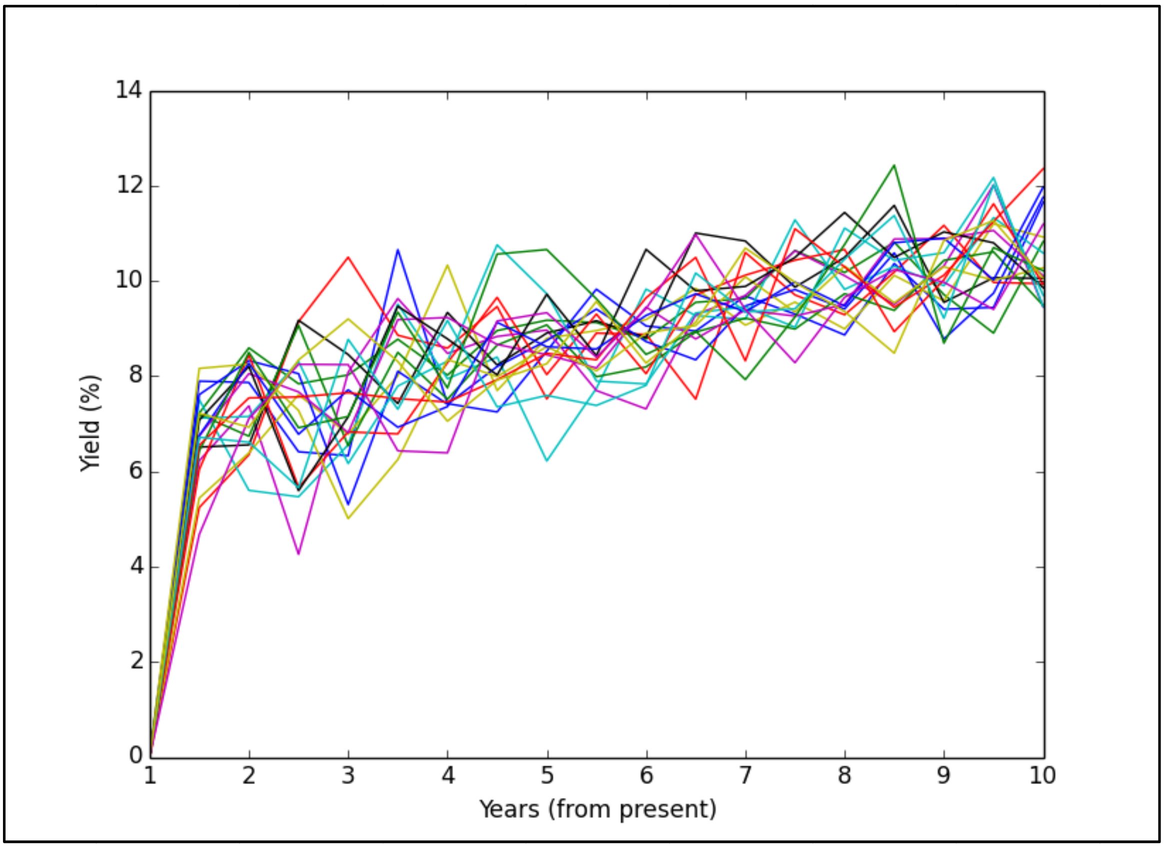


Figure 3-2: 20 possible term structures that could be observed one year from the present

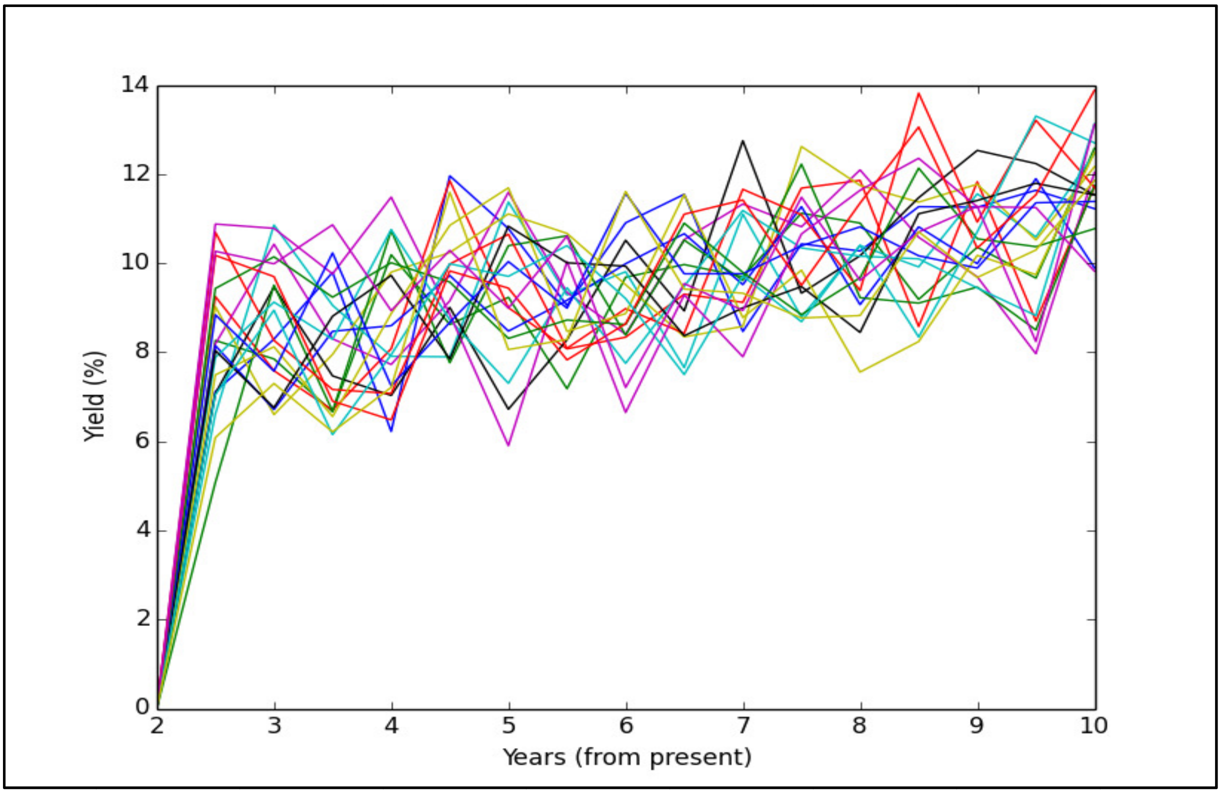


Figure 3-3: 20 possible term structures that could be observed 2 years from the present

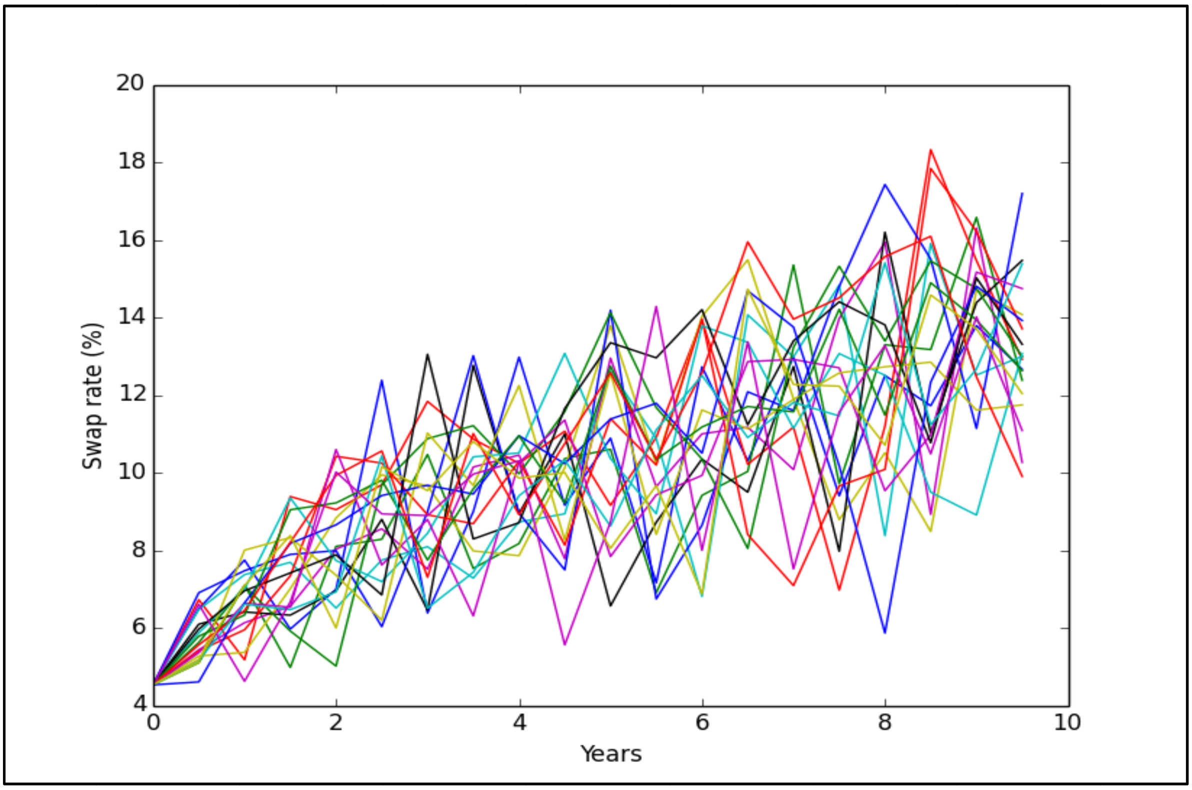


Figure 3-4: 20 possible swap rates that could be observed at each time from the present up to 10 years in the future

Figure 3-4 shows how many different possible swap rates could be observed in the future, for interest rate swaps that end on the date (ten years from the present). All these swaps therefore represent the possible costs of replacing a swap that was defaulted on by a counterparty. If the term structures appear too volatile, consider Figure 3-5, a graph of 10 year government bond yield for different specified countries. (Ro, 2014)

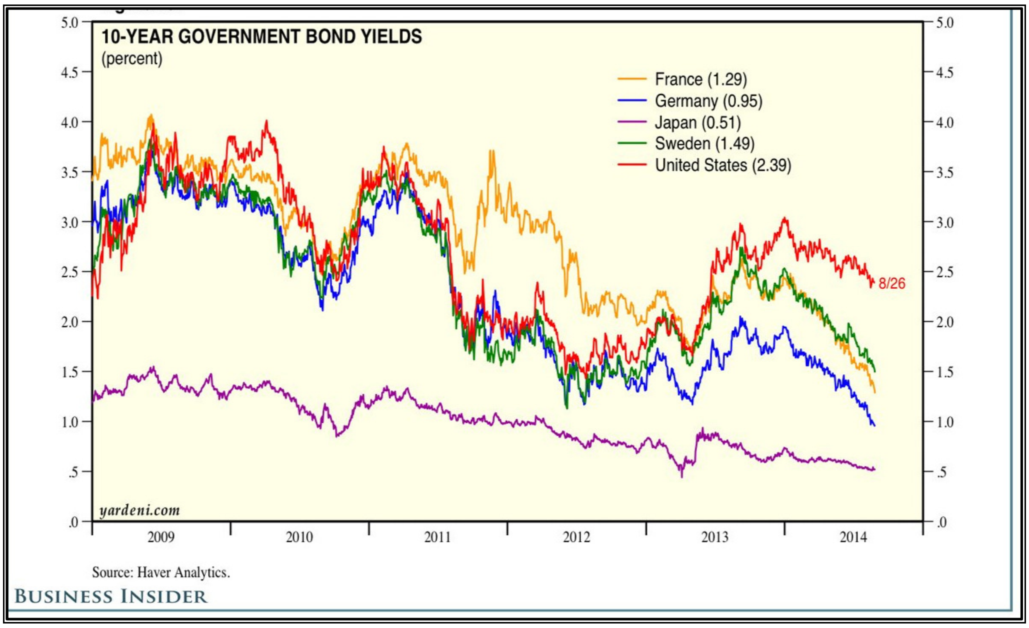


Figure 3-5: Historical 10-Year Government bond yields for various specified countries

In Figure 3-5, it can be seen that a 10 year government bond can vary considerably in yield over the course of just a few years. Figure 3-5 does not even take into account the big jumps in interest rates that occurred during the 2007/2008 financial crisis and even more at the start of the 80's, as shown below (Quadrini and Wright, 2012).

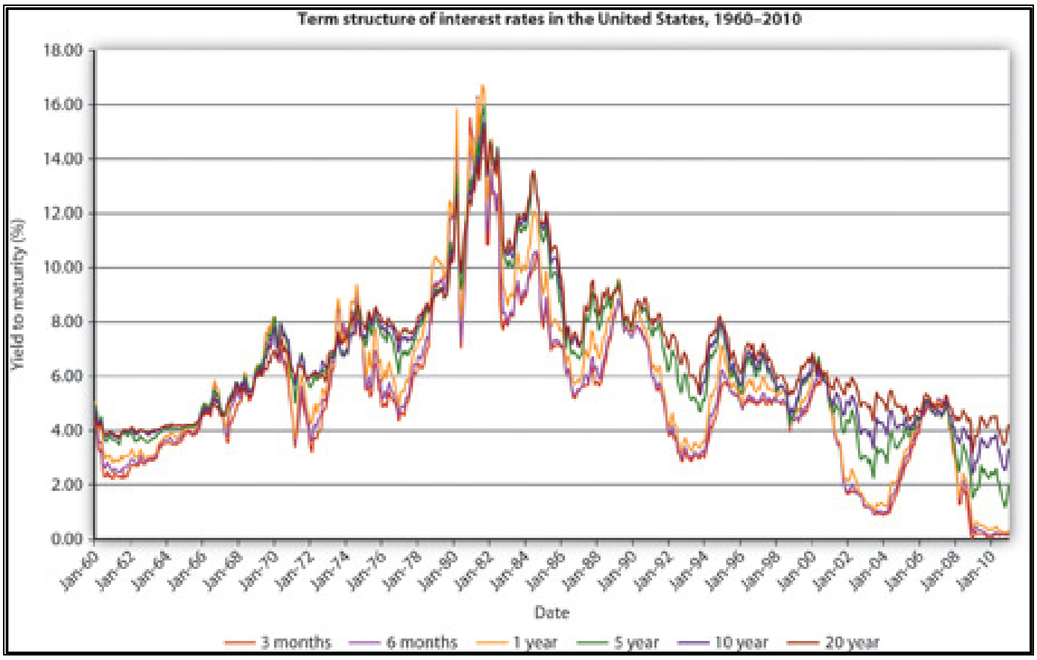


Figure 3-6: Observed US term structures between 1960 and 2010

It is therefore clear that market expectations about future interest rates can change considerably over relatively short periods of time, thus swap rates tend to vary considerably over time too, as shown below (van Dalen et al. 2014)

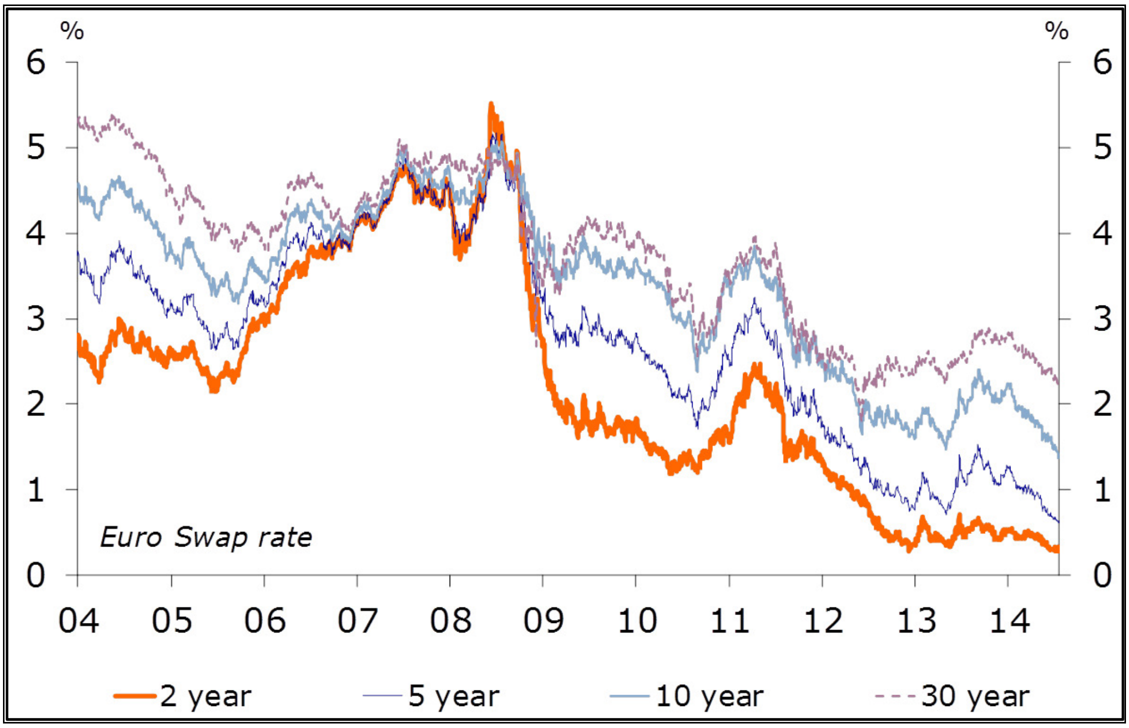


Figure 3-7: Historical swap rates for specified terms

If swap rates can vary considerably over time, then a simple mark-to-market assessment of CVA will also vary considerably. This is because swap rates on a certain date only takes into account the market's expectations of interest rates on that date, as the swap rate formula in equation (5) implies, and not how those expectations might change over time. The Hull-White one-factor model is able to simulate different term structures, and thus different market expectations, so using it to model CVA is more appropriate. Even better, using a swaption price calculated under the Hull-White model takes all the expectations into one neat number, since a swaption price is an expectation of discounted future swap rates at a certain date (defining the future swap rate as a random variable) as implied by equation (10) below.

$$\boldsymbol{P_{swaption}} = \boldsymbol{\xi(0)E_0}\left[\frac{\mathbf{max}(\boldsymbol{S(t)}, \mathbf{0})}{\boldsymbol{\xi(t)}}\right]$$

Note that Figure 3-5 also shows the importance of calibrating the model, since Japanese interest rates, for example, shows considerably different behaviour from German and French interest rates.

4 Research methodology

In this chapter, the process followed in obtaining the results in the next chapter will be outlined from a practical perspective. This will include rationale behind the decision on the models used, how the data required was obtained and processed, calibration of each model, and finally how the results were calculated.

4.1 Models

For the pricing of European swaptions, Black's Forward Rate model was be used because of its popularity in pricing interest rate derivatives (Hull, 2012, p.664), and its simplicity. The downside of Black's model, however, is that it does not provide a description of how interest rates evolve through time (Hull, 2012, p.682). Therefore each swaption was calculated using Hull-White's model as well. An analysis of the differencing in CVA under the two models will then be possible in section 5, the results section.

4.2 Data

For the calculation of CVA using the swaption approach, it was assumed that loss given default (LGD) was equal to one (i.e. recovery rate equal to zero). Data collected for the CVA calculations included a matrix of swaption implied volatilities and swap rates for the relevant dates. The dates that the data was relevant for are 20, 21 and 27 February 2014, 26 March 2014, and 19 June 2014. The spacing of the dates is specifically to illustrate how the CVA and EE profile evolve over long and short periods.

Unfortunately, a zero curve for the South African market could not be obtained. Instead, 3 month JIBAR rates up until 18 months were used, and for yields over longer periods than that, yields on coupon bearing government bonds, instead of zero coupon bonds, were directly used. This might cause the yield curve to be slightly too low, but it was thought that government bond yields would still provide a decent approximation as to how the zero curve would evolve through time.

Observed market swap rates were obtained for each of the dates that CVA was assessed on. These were used to calculate CVA using the MtM approach. The exact method used for this calculation in given in section 4.6. Note that each of these swap rates were for swaps with a maturity of ten years. Each of the market swap rates obtained for subsequent dates therefore described a swap with a different maturity date as the swap held. However, considering that the swap held had a maturity of 10 years, and the greatest difference in maturity dates from the original swap was about a third of a year, these differences were not considered to be problematic. It was considered to be the best approximation to market swap rates for the swap held. The alternative was to use the term structure to obtain a market rate which would not have been entirely accurate either. This is because the term structure was not a true zero curve, as explained above.

The data mentioned here was obtained from Rand Merchant Bank (2014).

4.3 Calibration

Firstly, the Hull-White model was calibrated to the South African market. This included using the techniques discussed in section 2.4.6, on model calibration. Black's model was calibrated by using volatilities in the implied volatility matrix, and interpolating where necessary. Calibration is covered in more detail in sections 4.3.1 and 4.3.2. After the Hull-White one-factor and Black's model were calibrated, swaption prices relevant to the South African market could then be calculated to find appropriate values for CVA.

For both models, an implied volatility matrix for at-the-money swaptions was used. An example of such a matrix is shown below

Maturity		Tenor				
	3 Months	6 Months	1Y	4Y	10Y	20Y
6 Months	0.25	0.23	0.22	0.2	0.19	0.18
1Y	0.25	0.24	0.23	0.21	0.20	0.19
4Y	0.26	0.25	0.24	0.22	0.22	0.19
10Y	0.24	0.24	0.23	0.215	0.23	0.2
20Y	0.24	0.23	0.22	0.21	0.2	0.18

Table 4-1: Example of Implied volatility matrix for at-the-money swaptions used for calibration

Note in the table above that maturity dates for the swaptions are listed in the left-hand column, while tenors (or swap lifetimes) for each underlying swap are listed in the row on top of the highlighted matrix. The intersection of the second row and fourth column in the matrix, for example, gives the implied volatility for a swaption with a maturity of one year and a tenor of four years.

4.3.1 Hull-White model

The parameters a and σ were estimated using the methods described in section 2.4.6, using a least squares optimisation function provided by the scipy package (an add-on to the Python language). Initial guesses for a and σ were first obtained by using methods 1 and 2, then method 3 was applied to find estimates for a and σ simultaneously.

Note that only implied volatilities for co-terminal swaptions to the swaptions to be valued (benchmark swaptions) were used. Co-terminal swaptions have swaps that have the same maturity date, as underlying, that is, the sum of the maturity and tenor (from the matrix) are the same. Swaptions were included if they differed by less than 30% of the time from the present to the terminal date (date of last payment) of the benchmark swaptions. As an example, if the benchmark swaptions had its terminal date 10 years from the present, implied volatilities of swaptions with terminal dates between 7 and 13 from the present were had a weight of 1 in equation (33). The rest had a weight of zero.

Calibrating the model with this methodology was computationally intensive, but provided the model that was the most accurate reflection of the real-world interest rate environment. The code used for the calibration is in Appendix C.

4.3.2 Black's model

Implied volatilities are generally obtained by inverting Black's model to market swaption prices. Calibration for Black's model was therefore relatively simple.

If a swaption to be valued matched an existing tenor and maturity in the implied volatility matrix, that exact volatility would be used for the swaption. Where only one of the maturity or tenor of the swaption matched a value in the matrix, linear interpolation was used to find the missing value (in this case maturity) with the formula

$$IV_{interp} = \frac{IV(t_1, u)(t - t_2) + IV(t_2, u)(t_1 - t)}{t_2 - t_1}$$

where

- IV_{interp} is the implied volatility that is to be interpolated
- $IV(t, u)$ is the implied volatility associated with date maturity date t and tenor u
- t is the date that we are interpolating on (the date that is not in the matrix) with $t_1 < t < t_2$ the closest dates that we have maturities on in the matrix

In the formula above, if we interpolated for tenor, we would have kept t constant, and interpolated for u.

Where neither maturity t, with closest maturities $t_1 < t < t_2$, nor tenor u with closest tenors $u_1 < u < u_2$, could be found in the matrix, bilinear interpolation was implemented by first interpolating for maturity, for the two tenors either side of the desired tenor:

$$Mat^{(1)}{}_{interp} = IV_{interp} = \frac{IV(t_1, u_1)(t - t_2) + IV(t_2, u_1)(t_1 - t)}{t_2 - t_1}$$

$$Mat^{(2)}{}_{interp} = IV_{interp} = \frac{IV(t_1, u_2)(t - t_2) + IV(t_2, u_2)(t_1 - t)}{t_2 - t_1}$$

Where

- $Mat^{(1)}{}_{interp}$ is the value of the implied volatility for the maturity interpolated for tenor u_1
- $Mat^{(2)}{}_{interp}$ is the value of the implied volatility for the maturity interpolated for tenor u_2

The required implied volatility was then interpolated as

$$IV_{interp} = \frac{Mat^{(1)}{}_{interp}(u_2 - u) + Mat^{(2)}{}_{interp}(u - u_1)}{u_2 - u_1}$$

It must be noted that the calculated volatility for Black's model using this method is time-dependent, since a different value for sigma is calculated for each increment in time.

4.4 Calculation of EE and CVA

In the calculation of the EE profile, two approaches were followed, namely using the Hull-White and Black's model. In both approaches, a year was assumed to contain 250 (business) days. In the absence of any collateral (the first approach), EE was calculated on a weekly basis, regardless of cash flow frequency. This meant that every 5 business days EE was calculated, so a total of 50 swaptions were valued in a year. Although this sacrificed a small amount of accuracy in the calculation, it was well worth the reduction in time required to calculate EE values for each business day. To illustrate, consider the CVA values and time required for the calculation of a 20 year swap under the following circumstances

Model	Frequency	Time (seconds)	CVA	% difference in CVA
Black	Daily	173.17	126.08	0.238
	Weekly	34.17	125.78	
Hull-White	Daily	326.51	64.82	0.108
	Weekly	66.76	64.75	

Table 4-2: Comparison of calculation speeds for daily and weekly frequencies

It can be seen from the results above that calculating CVA based on EE values spaced 1 week apart instead of daily EE values brings massive benefits in terms of speed of calculation, while sacrificing very little in terms of accuracy, less than 0.3% of the more accurate (daily) calculation, in fact.

After all the EE values were calculated, CVA was calculated by using equation (7). The integral was approximated by the following numerical integration algorithm:

$$\sum_{i=1}^{n}(T_i - T_{i-1})\frac{EE(T_i) + EE(T_{i-1})}{2}$$

The exact code used for this algorithm can be found in Appendix C.

4.5 Calculation of CVA DV01

Finally, a DV01 measure for CVA was also calculated. In the case of the Hull-White model, for each bond, this was calculated by measuring the change in the bond's value for a very small drop in bond yield. In this way, a derivative w.r.t. a basis point shift in the zero curve was estimated by modifying the forward difference, defined in section 2.3.4. For Black's model, a simpler method was used.

4.5.1 Hull-White model

The formula for the delta of a payer swaption was given in section 2.3.7 as

$$-\frac{\partial P_{HW}(0,T,T_M)}{\partial w(bp)} = \sum_{i=0}^{n} c_i N(-Z^* - v_i)\frac{\partial D(0,t_i)}{\partial w(bp)} \tag{27}$$

Since equation (27) is only for an ordinary European swaption, it needs to be modified to work on a tail swaption. Therefore equation (27) becomes

$$\begin{aligned}-\frac{\partial P_{HW\,tail}(0,T,T_M)}{\partial w(bp)} &= -\frac{\partial D(0,0,T_0)}{\partial w(bp)}N(-Z^{**} + \bar{v} - v)\exp\left(v\bar{v} - \bar{v}^2\right)\\ &+ \frac{\partial D(0,0,T)}{\partial w(bp)}N(-Z^{**}) + K(T-T_0)\frac{\partial D(0,0,T_1)}{\partial w(bp)}N(-Z^{**} - v)\\ &- \frac{\partial P^*_{HW}(0,T,T_M)}{\partial w(bp)}\end{aligned} \tag{46}$$

where $-\frac{\partial P^{*}_{HW}(0,T,T_M)}{\partial w(bp)}$ is equation (27), except that Z^{*} gets replaced by Z^{**}, which was defined in section 2.3.2.

Since the formula for collateral agreements also contains the formula for no collateral agreements (under a tail swaption) we will work here exclusively with that formula, obtaining the DV01 formula for EE under no collateral in the process. The formula for EE under collateral agreements is given by

$$P_{collateral} = P_{HW\,tail} - P^{**}_{HW\,tail} + HN(-Z^{**}) \quad \textbf{(25)}$$

so it follows that

$$\frac{\partial P_{collateral}}{\partial w(bp)} = \frac{\partial P_{HW\,tail}}{\partial w(bp)} - \frac{\partial P^{**}_{HW\,tail}}{\partial w(bp)}$$

where $-\frac{\partial P^{**}_{HW\,tail}}{\partial w(bp)}$ is equation (46), except that Z^{**} gets replaced by Z^{***}, as defined in section 2.3.3.

In order to calculate $\frac{\partial P_{HW\,tail}}{\partial w(bp)}$ and $\frac{\partial P^{**}_{HW\,tail}}{\partial w(bp)}$, a numerical approximation was used. From section 2.3.5, we use

$$\frac{dg}{dx}(0.0001) = DV01 = \frac{g(x_0 + h) - g(x_0)}{h}(0.0001) \quad \textbf{(29)}$$

more specifically,

$$\frac{\partial D(0,t_i)}{\partial w(bp)} \approx -\frac{\partial D(0,t_i)}{\partial w(h)}(0.0001) = DV01 = \frac{D_t(\overline{x_0} + h) - D_t(\overline{x_0})}{h}(0.0001)$$

where

- $D_t(\overline{x_0})$ is the price of a discount bond maturing at time t, given a vector that represents zero curve rates $\overline{x_0}$
- h is a very small number, e.g. 10^{-8}

Therefore in the algorithm used to calculate this, the difference between two zero-coupon bonds, one on a zero curve with rates decreased by h, and the other on the original zero curve, was calculated then divided by h, then multiplied by 0.0001 to get the change in price for a basis point. After $\frac{\partial D(0,t_i)}{\partial w(bp)}$ has been approximated, it will be used to calculate $\frac{\partial P_{HW\,tail}}{\partial w(bp)}$, $\frac{\partial P^{**}_{HW\,tail}}{\partial w(bp)}$ and also $\frac{\partial P_{collateral}}{\partial w(bp)}$.

For a receiver swaption, the same methodology is followed, except that we start with equation (28) instead of equation (27).

The exact code used for this algorithm can be found in Appendix C.

4.5.2 Black's model

The DV01 for a swaption calculated under Black's model was computed by a simpler numerical method. Since DV01 is the change in price for a basis point increase for all interest rates (see section 1.3), the DV01 could be approximated by

$$\frac{\partial \boldsymbol{P_B}}{\partial \boldsymbol{w}(\boldsymbol{bp})} \approx (\boldsymbol{P_+} - \boldsymbol{P_-})/\boldsymbol{2}$$

Where P_+ and P_- are the swaption prices for an upward and downward shift in the yield curve, respectively. The downside of this algorithm is that the swaption price is calculated twice for each date required, instead of once for each date as in the Hull-White DV01 case. On the other hand, since the algorithm for calculating a swaption price under Black's model is computationally less demanding than under the Hull-White model, this should not pose a significant problem.

The code for this algorithm is also in Appendix C.

4.5.3 Obtaining CVA DV01

When the DV01 for each swaption has been found, as shown in the previous two subsections, CVA DV01 is found by integrating as in section 4.4, except that EE values are replaced by their respective swaption DV01 values. This is done in the same way for each model.

4.6 Calculation of swap market values

For the purposes of comparing CVA obtained under the semi-analytical approach with values obtained using the MtM approach, market swap values were calculated by assuming that the market swap rate on any date complied with equation (5):

$$C(0,0,n) = \frac{1 - D(0,0,n)}{\sum_{i=1}^{mn} D(0,0,\frac{\mathrm{i}}{\mathrm{m}})h} \quad \textbf{(5)}$$

This implies that $\boldsymbol{B_{new\,fixed}} = \boldsymbol{B_{new\,floating}}$ in

$$\boldsymbol{S_{new}}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{n}) = \boldsymbol{B_{new\,fixed}} - \boldsymbol{B_{new\,floating}} \quad \textbf{(1)}$$

where a newly issued swap, $\boldsymbol{S_{new}}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{n}) = \boldsymbol{0}$**,** since we are assuming the observed market swap rates are such that the swap value is zero at inception. Since $\boldsymbol{B_{new\,floating}} = \boldsymbol{B_{floating}}$ in the newly issued swap and the swap currently held,

$$\boldsymbol{S}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{n}) = \boldsymbol{B_{fixed}} - \boldsymbol{B_{floating}}$$

We have

$$\boldsymbol{S}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{n}) = \boldsymbol{B_{fixed}} - \boldsymbol{B_{new\,fixed}}$$

With the assumptions made, we can therefore calculate the market swap value as

$$\boldsymbol{S}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{n}) = \boldsymbol{A}(\boldsymbol{0})(\boldsymbol{C_{fixed}} - \boldsymbol{C_{new\,fixed}})$$

where C_{fixed} and $C_{new\,fixed}$ are the swap rates for the swap currently held and the observed market swap rate, respectively.

5 Results

The first section of the results will deal with each of the models themselves and how they react to different inputs, specifically term structures, parameters, collateral and netting agreements. Explanations for how the models react to inputs will be given where possible. Fictional term structures were used in the first section, and no implied volatility matrices were used so that inputs could be controlled better. The following simple term structures, labelled term structures 1, 2 and 3, in order from left to right, will be used:

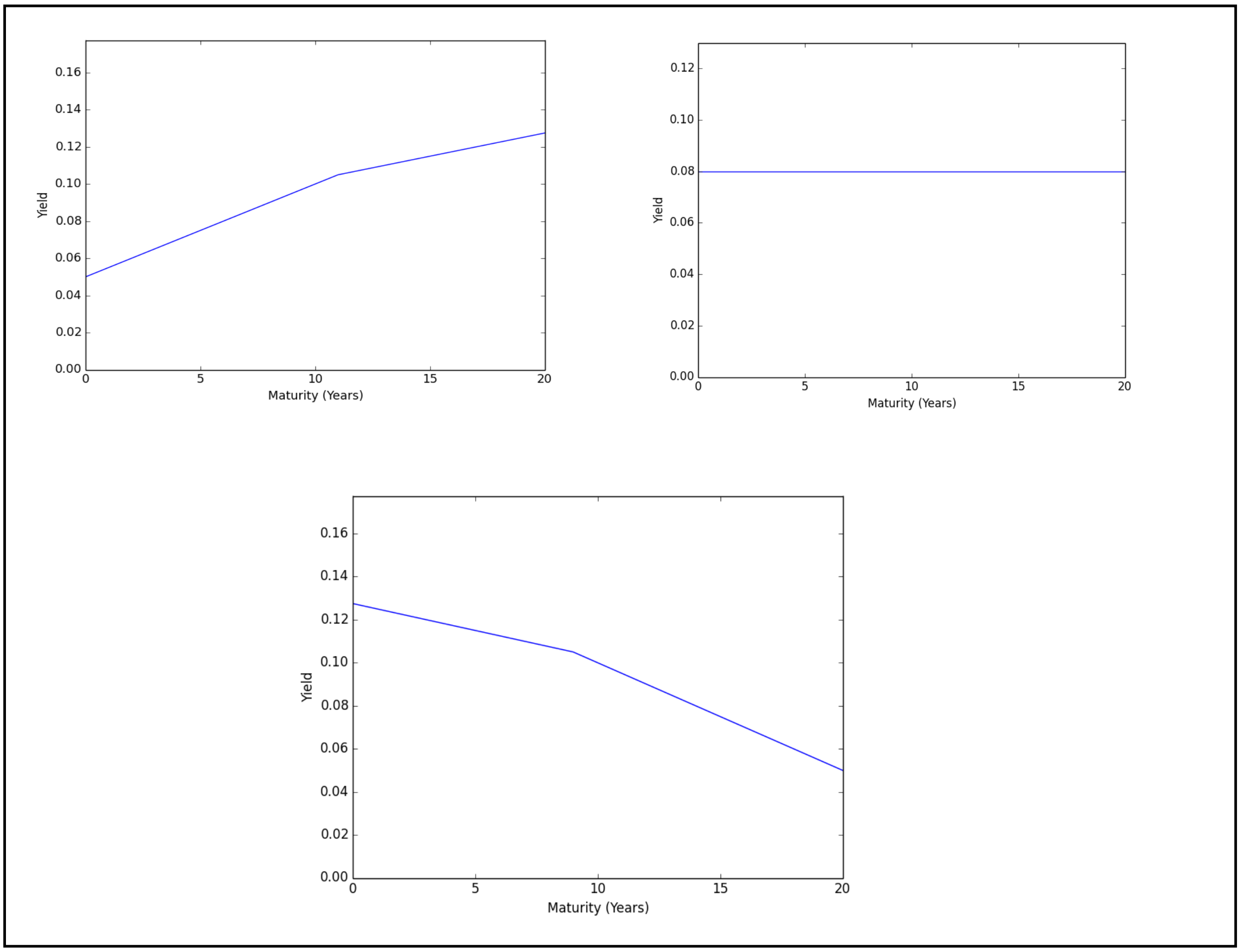


Figure 5-1: Term structures 1, 2, and 3. Each term structure will show the effect that a rising, flat and declining term structure has on the EE profile

Each time, EE and CVA were calculated for a portfolio of one swap only, with a maturity of 10 years and a notional value of R100 000. Payments are semi-annual for both legs of the swap. The single swap was used so that the effects of netting would not affect the results. Netting was handled separately. Where factors other than the term structures were tested, the rising term structure was always used for consistency. The effect of term structure was only tested under Black's model, since it was thought that it would have the same effect under both models. In Black's case, where volatility was not explicitly tested, 15% volatility was used. Note that in every case given in section 5.1 and 5.2, the market value for the underlying swap was zero.

Note that CVA values calculated in sections 5.1 and 5.2 under the different models should not be compared. Only in section 5.3, where both models were calibrated to the same market data, were CVA and exposure profiles compared.

5.1 Black's model

In this section, results for the fictional inputs will be shown and discussed under Black's model. It is divided according to each factor that was tested, and then divided further into results for payer and receiver swaps.

5.1.1 The effect of volatility

5.1.1.1 Payer swaps

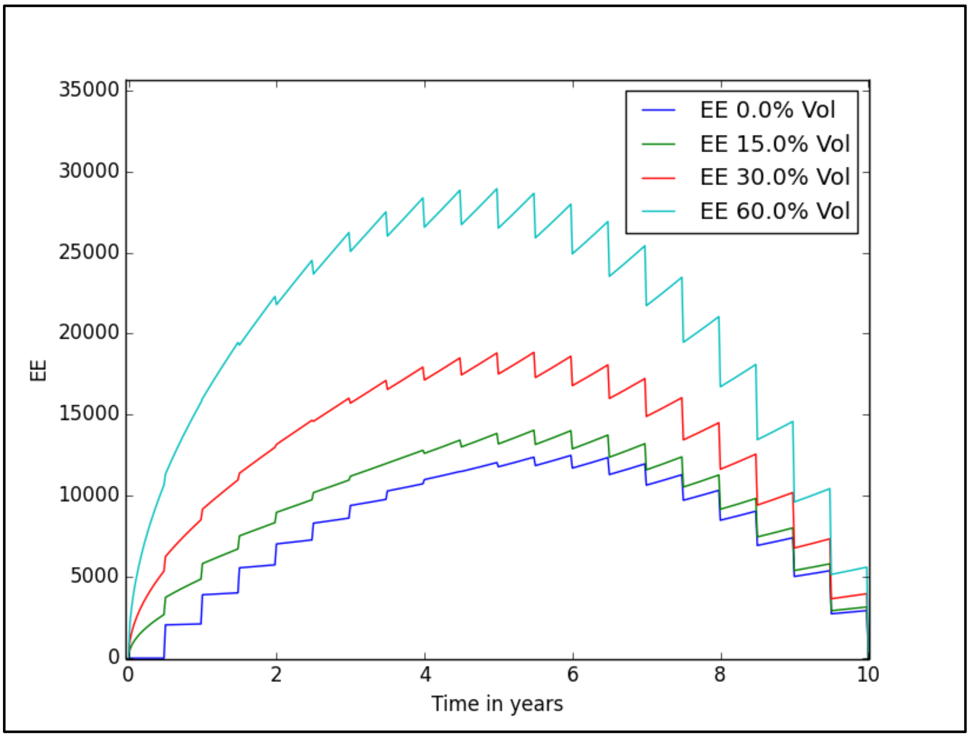


Figure 5-2: Exposure profiles for different volatilities for a payer swap under Black's model

It can be seen that the payer swap's exposure, above, does not change shape much, but rather scales upward as volatility increases. The CVA value, shown below, increases with volatility, as expected. Interestingly, the DV01 measure decreases as volatility and CVA increases.

Sigma	**CVA**	**DV01**
0%	5 391.84	24.49
15%	6 518.40	17.81
30%	9 087.33	14.61
60%	14 333.84	12.01

Table 5-1: Numerical Results for exposure calculations

5.1.1.2 Receiver swaps

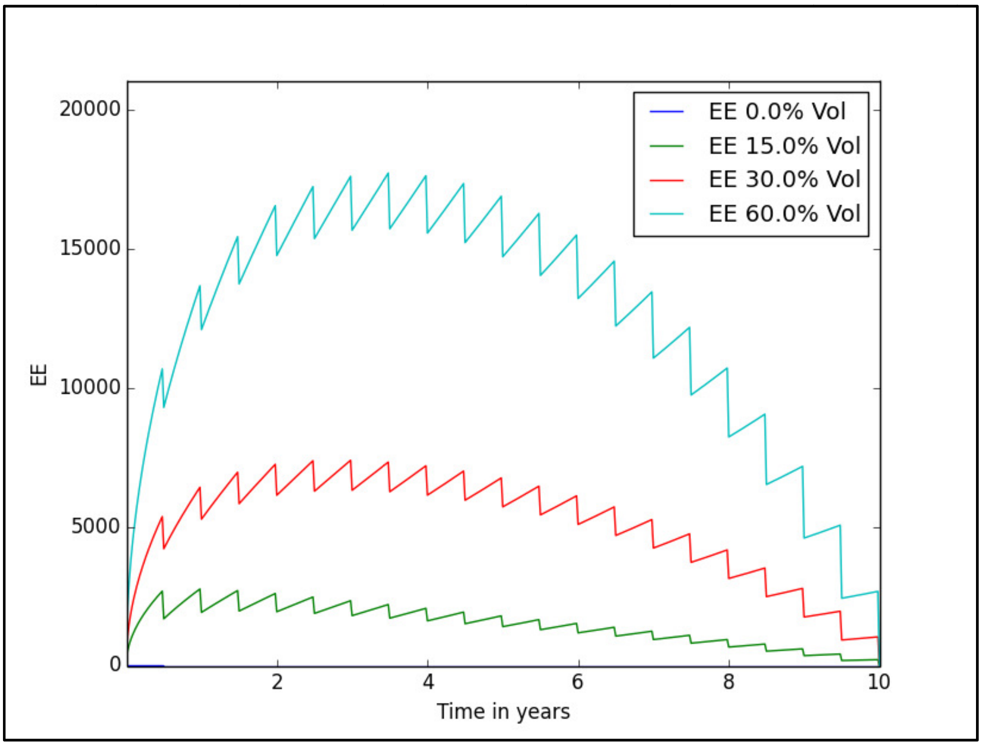


Figure 5-3: Exposure profiles for different volatilities for a receiver swap under Black's model

The receiver swap's exposure also scales upward as volatility increases, but also changes shape slightly more: it becomes more humped towards the top, although that could partly be due to the much larger differences in exposure profiles from changes in volatility. This is reflected in CVA values: CVA more than doubles every time volatility is pushed upward by 15%. Absolute DV01 values increase, rather than decrease as CVA and volatility increase, in this instance.

Sigma	CVA	DV01
0%	0.00	-1.59
15%	1 126.56	-8.27
30%	3 695.49	-11.47
60%	8 942.00	-14.07

Table 5-2: Numerical results for exposure calculations

5.1.2 Effect of the term structure

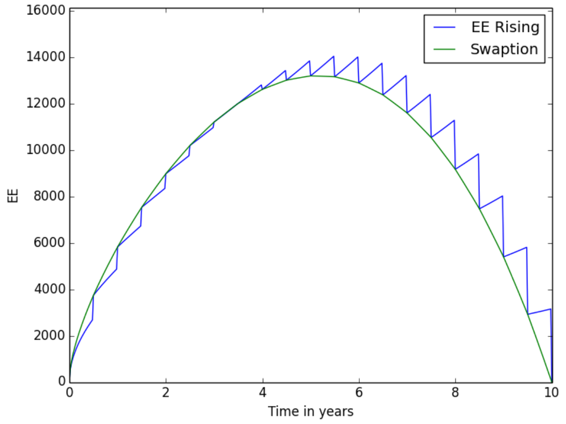
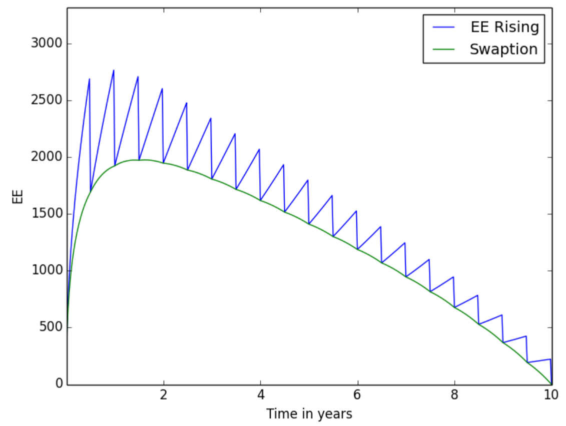
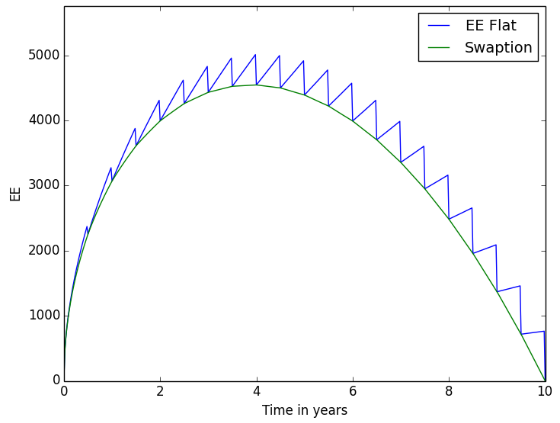
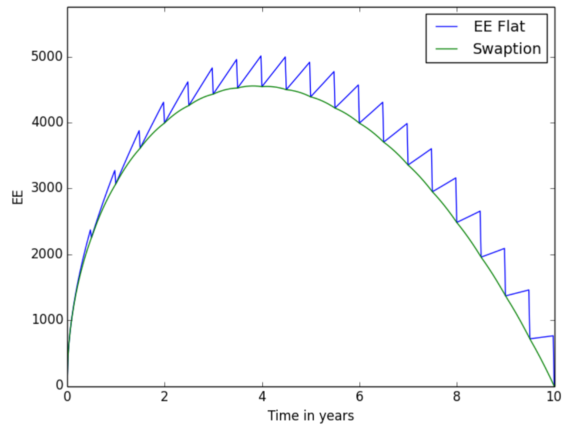
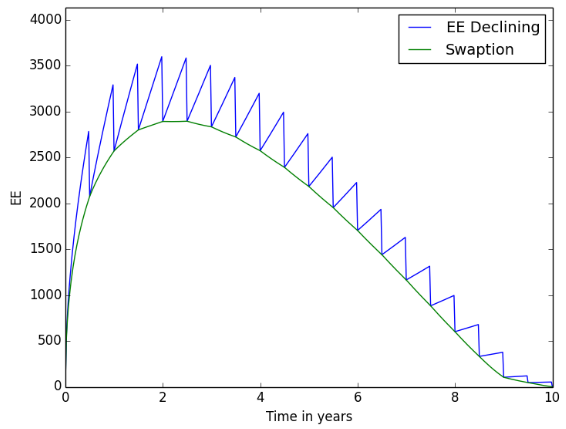
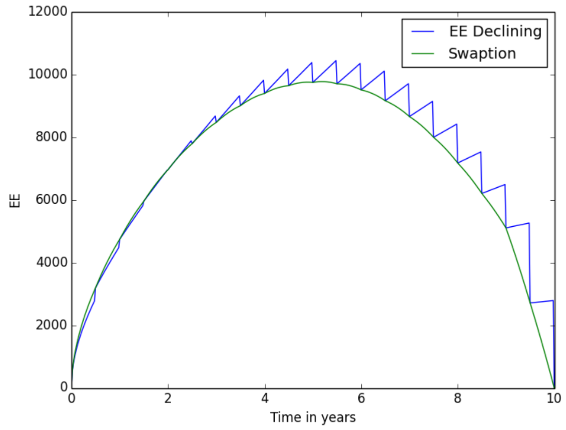
Payer swap	Receiver swap
	
	
	

Table 5-3: Left column: Effect on EE profile of changing the term structure from rising, to flat, to declining for a payer swap. Right column: Effect on EE profile of changing the term structure in the same way for a receiver swap

Payer swap		
Term structure	**CVA**	**CVA DV01**
Rising	6 518.40	17.81
Flat	2 461.16	15.95
Falling	1 337.50	10.29

Table 5-4: Numerical results for the term structure effects in Table 5-3 in the case of a payer swap

Receiver swap		
Term structure	**CVA**	**CVA DV01**
Rising	1 126.56	-8.27
Flat	2 461.18	-16.22
Falling	4 333.18	-19.83

Table 5-5: Numerical results for the term structure effects in Table 5-3 in the case of a receiver swap

Exposures were plotted on different sets of axes this time, since the different profiles for different term structures overlapped considerably. In order from top to bottom, the table on the previous page lists payer and receiver swap exposure profiles for a rising, flat and falling term structure. The payer swap profile leans to the right in a rising term structure environment, and slants gradually to the left and falls as the term structure becomes declining. The opposite is true for receiver swaps. The profile of a payer swap in a rising interest rate environment is very similar to the profile of a receiver swap in a falling interest rate environment, and vice versa.

CVA values for a payer swap decreased as the term structure became more downward sloping, while at the same time increasing for a receiver swap. This can be understood by considering the concept of net present value (NPV). If the term structure is upward sloping, the swap rate for the first portion of the swap is more than the forward rates. This means that parties expect that the party holding the payer swap will have negative cash flows for the first portion of the swap and positive cash flows for the second portion. The party holding the payer swap therefore expects to get compensated later during the second portion of the swap. That party carries the exposure on average, since if the other party defaults, that party will not get compensated. As the term structure becomes more downward sloping, this effect diminishes.

If a change in the term structure shape is the cause of the change in CVA, CVA DV01 values imply that exposure to interest rate movements increases with CVA.

On a side note, the graphs on the previous page illustrate the adjustment made for payments lost during the default period (see sections 2.2.2, 2.3.2 and 2.4.2). The green line in each graph represent an ordinary European swaption, maturing at that time with an underlying swap maturing 10 years from the present time, while each of the jagged blue lines represent the “tail swap” after making the adjustment described in section 2.2.2. Under lower volatilities, the first adjustments are still negative in the payer swap case, reflecting the negative cash flows in the first portion of the swap contract in a rising interest rate term structure environment (the first graph, top left). As volatility increases, the adjustment becomes positive, so then the “spikes” will all be on top of the swaption curve.

5.1.3 Collateral agreements

In each of the following cases, the threshold in the collateral agreement was at R1 000, 1% of the notional value of the swap.

5.1.3.1 Payer swap

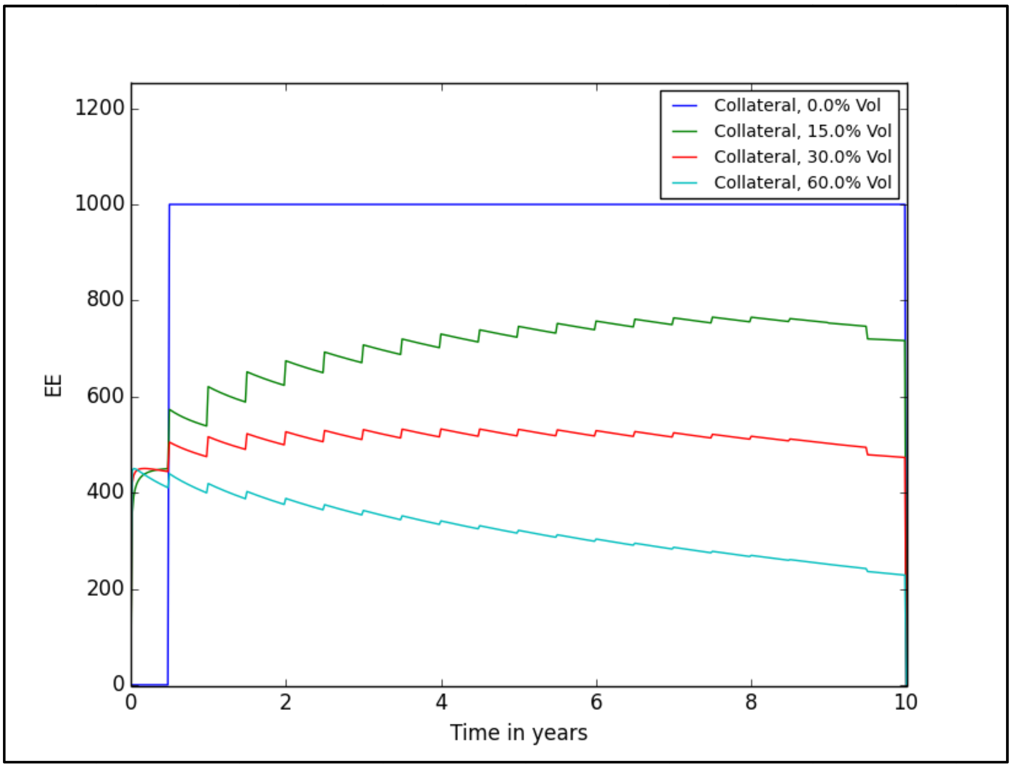


Figure 5-4: Exposure profiles under various volatilities for a payer swap under a collateral agreement

The profile of the payer swap under zero volatility is explained by considering NPV once again. The NPV plot shows the swap value (exposure if positive) if swap rates never change, that is, if volatility is zero (see Figure 5-5). From this fact, it is easy to see how the exposure would be constantly equal to the threshold if it is low enough in relation to the exposure profile. If volatility is zero, the future exposure becomes deterministic, therefore it would “definitely” be higher than the threshold at all times in the future, in the payer swap’s case. In the first half-year of the payer swap, the exposure is expected to be zero because the first payment will be a loss, therefore it would be no loss to the party holding the payer swap to lose that payment. In the case of the receiver swap with a collateral agreement, zero volatility still implies a non-existent profile if the term structure is rising.

As volatility increases, the profile of the payer swap actually declines and falls, surprisingly. This is explained by considering that the payer swap already has exposure under zero volatility, as shown in

Figure 5-5, left. Introducing volatility then introduces more possibilities for the exposure to be lower than the threshold, reducing EE.

Sigma	CVA	DV01
0%	637.85	1.30
15%	465.56	0.90
30%	349.13	0.47
60%	232.92	0.22

Table 5-6: CVA decreased in accord with EE profiles, as well as DV01

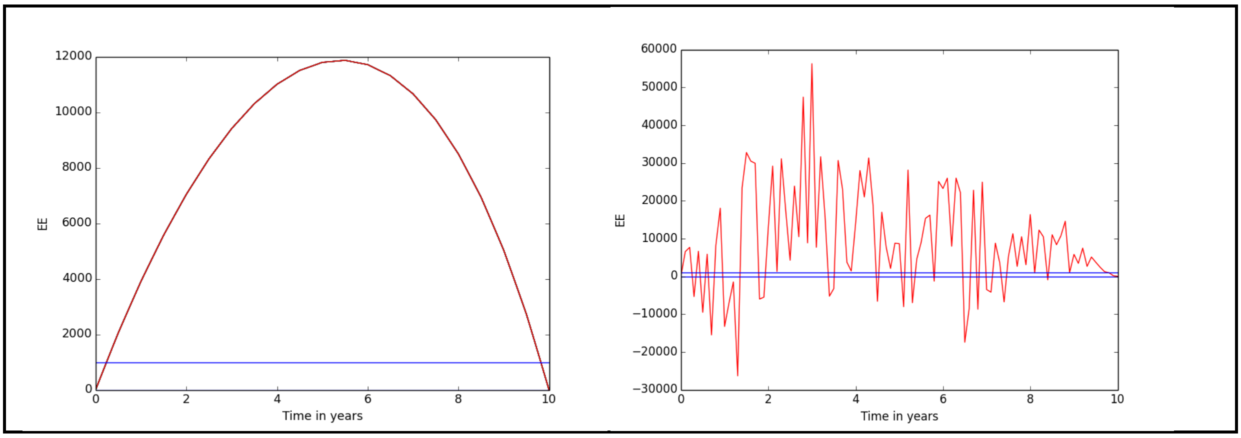


Figure 5-5: Left: Simulated price path for a payer swap under zero volatility.
Right: Simulated price path under low volatility. In both graphs, the upper blue line represents the collateral threshold, while the lower blue line represents zero.

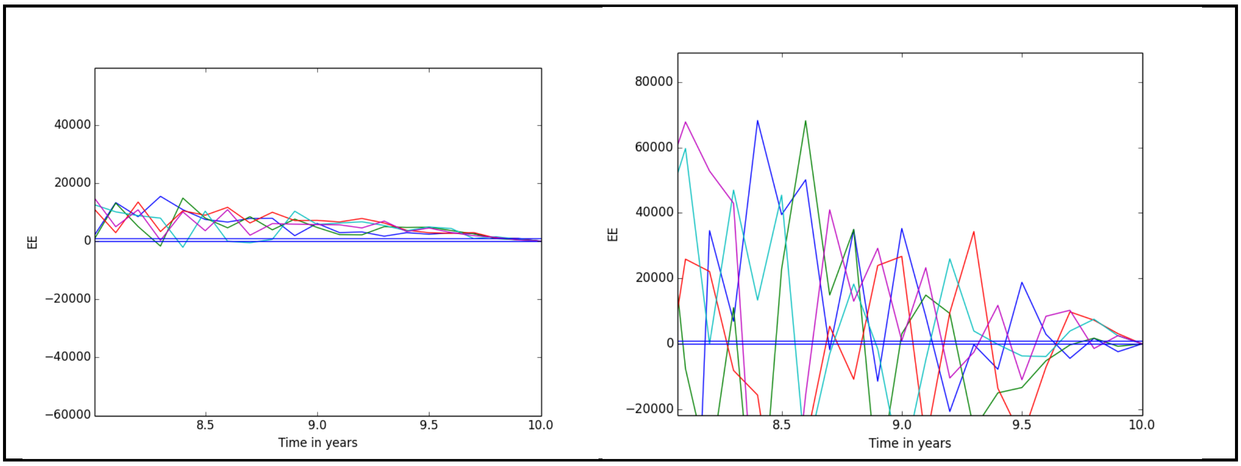


Figure 5-6: Left: Last 2 years of 5 simulated swap contract prices under low volatility.
Right: Last 2 years of 5 simulated swap contract prices under high volatility

Since the swap value depends directly on the swap rate, increasing volatility of the swap rate introduces more possibilities for the exposure (or swap value) to be less than the threshold, and equal to zero, as shown by the right-hand graph of Figure 5-5.

Under no collateral agreement, the increased possibilities of exposure being equal to zero (from increased volatility) is more than offset by the increased possibilities of higher exposure, thus pushing the exposure profile higher. Under a collateral agreement though, the possibilities of higher exposure with higher volatility are taken away and replaced only by possibilities of an exposure equal to the collateral threshold, while the possibilities where exposure is lower than the threshold, or zero, remain. Thus, as volatility increases, we have more possibilities for the swap value to be less than zero (therefore exposure is zero). This is not offset anymore by higher exposure possibilities, therefore exposure drops as volatility increases.

The gradual drop in EE as time passes is illustrated by the graphs of Figure 5-6. Even in the presence of volatility, in the last few years of a swap contract's life, it tends to have a more deterministic price, usually above the threshold. Introducing more volatility (shown in the right-hand graph) makes the swap's price vary more in those last few years too, pushing exposure down as explained.

5.1.3.2 Receiver swap

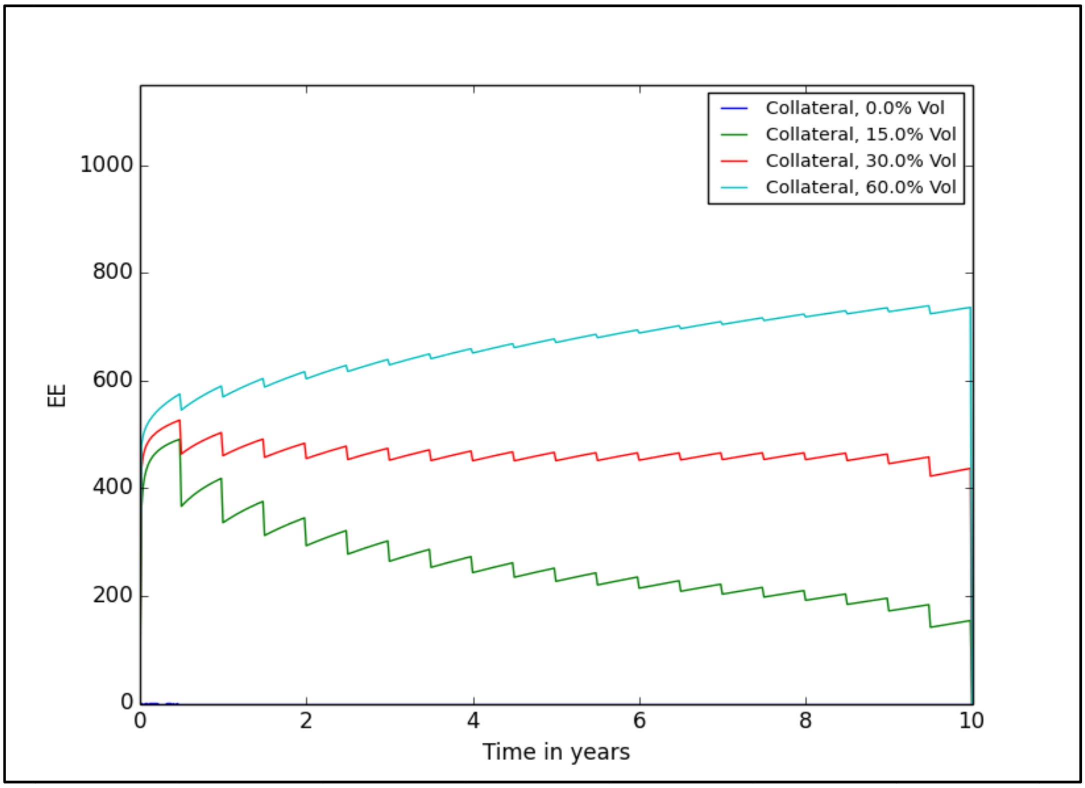


Figure 5-7: Exposure profiles under various volatilities for a receiver swap under a collateral agreement

In the case of a receiver swap, we know that NPV or price of a swap is the negative of the price of the payer swap. This implies that the net present value curve for the receiver swap is the payer swap NPV curve, but reflected along the x-axis. Therefore expected NPV is negative at every date. This implies that exposure is zero at every date, which implies the exposure profile in a rising term structure environment is non-existent under zero volatility. Therefore, increasing volatility increases possibilities for the exposure to be between zero and the threshold. As volatility increases, these possibilities keep on increasing, pushing the exposure profile up from zero.

Sigma	CVA	DV01
0%	0.00	-1.59
15%	191.24	-1.15
30%	319.21	-0.76
60%	444.66	-0.51

Table 5-7: In accord with EE profiles, CVA increased with volatility. DV01 came closer to zero as volatility increased, as in the payer swap case

In accordance with the exposure profiles, CVA decreases for a payer swap as volatility increases, and vice versa for a receiver swap. The CVA DV01 implies that both payer and receiver swaps became less sensitive to interest rate changes as volatility increased. This could be due to exposures being bounded from above and below in the case of collateral agreements. A drop in interest rates then, for example, would not matter at all if exposure was already well above the threshold or well below zero, since exposures are capped and floored, respectively, at the threshold and at zero.

5.1.3.3 Overall effect of collateral

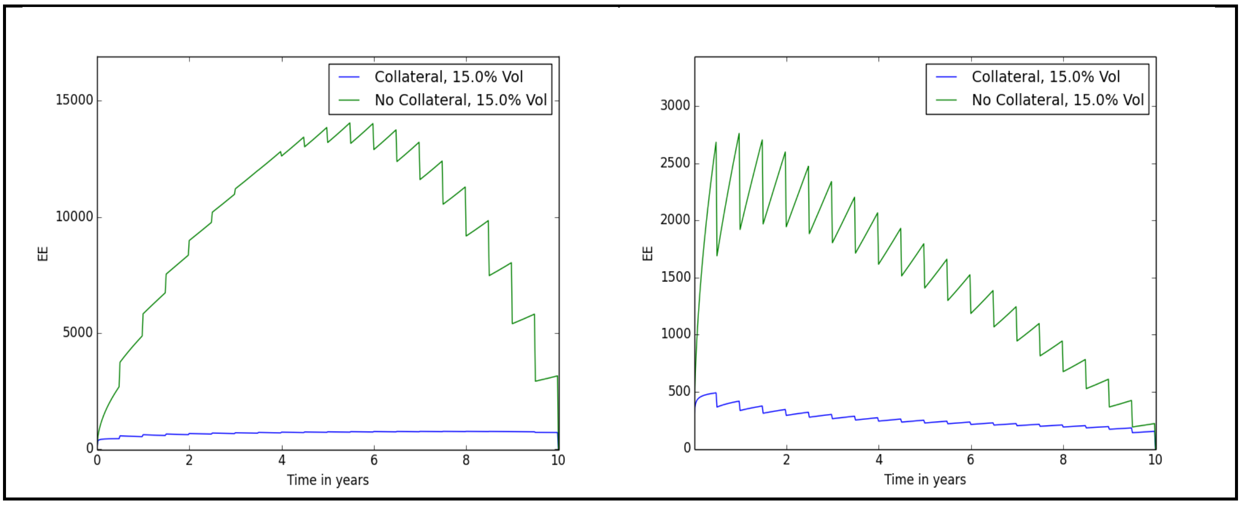


Figure 5-8: Left: The overall effect of a collateral agreement on exposure in a payer swap case. Right: Effect of a collateral agreement on exposure in a receiver swap case

The final graphs above show the effect that a collateral agreement has in the 15% volatility case. Note that the roles of payer and receiver swaps would have been reversed here if the term structure was falling, rather than rising.

5.1.4 The effect of netting

The table below shows the portfolio of swaps that was analysed for credit risk. All had the same swap rate (the same scenario shown in section 2.1.2.1).

	Start Date	End Date	Principal
Swap 1	11/10/2014	11/10/2024	100 000
Swap 2	11/10/2014	11/10/2021	100 000
Swap 3	11/10/2014	11/10/2018	200 000

Table 5-8: Netted swap portfolio

Figure 5-9 and Table 5-9, below, shows the EE profiles and numerical results for portfolios of payer and receiver swaps.

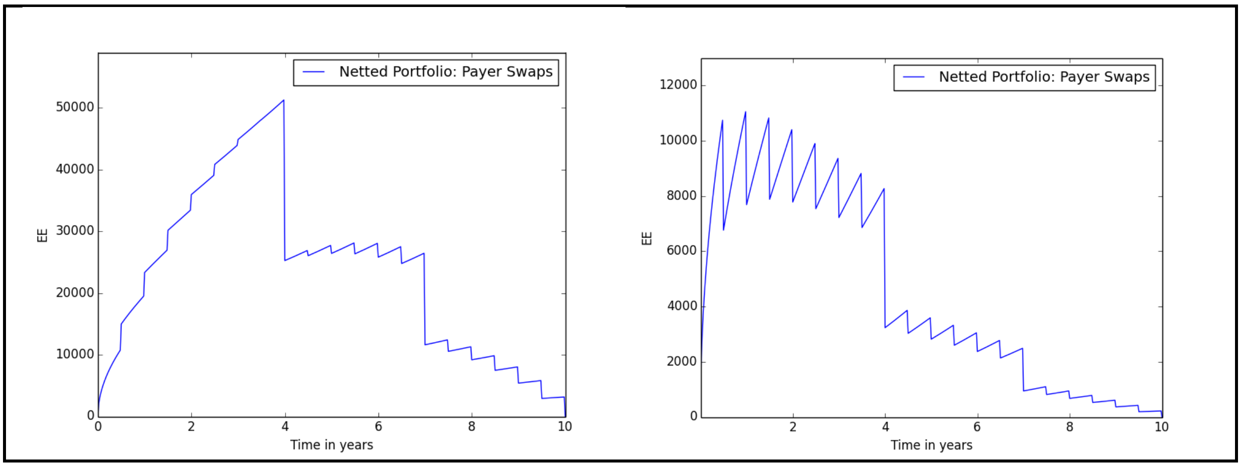


Figure 5-9 Left: Exposure profile for payer swap. Right: Exposure profile for receiver swap

	CVA	CVA DV01	MtM
Payer Swaps	17 364.72	59.54	-22 515.79
Receiver Swaps	3 649.72	-29.31	22 515.79

Table 5-9: Numerical results for the netted portfolios of payer and receiver swaps

One can see in each of the graphs in Figure 5-9 where a swap matured by the large downward jumps. CVA for the payer swap is much higher than that of the receiver swap because of the upward sloping term structure.

Note that the market values of the swaps in each of the above cases arise from the fact that all swaps had the same swap rate, but different maturities. Unless the term structure was flat, this implied that the NPV of some of the swaps will not be zero, causing the market value to be non-zero. It is interesting to note, however, that even though the market value is negative for the party holding the payer swap (party A), which should imply that party A carries no credit risk, the CVA values for the payer and receiver swaps imply that party A carries the larger credit risk. This is probably because the market value only takes into account one possible path of future interest rates (since it is deterministic), while the swaption approach takes into account all future potential paths of future interest rates.

5.2 The Hull-White model

Results for the fictional inputs under the Hull-White model are given here. The same format is followed as in the previous section.

5.2.1 The effect of the volatility parameter (sigma)

For the Hull-White model, changes in the exposure profile from changes in volatility reflect what was observed under Black's model. Exposure profiles with their respective CVA and DV01 derivations are given below.

5.2.1.1 Payer swap

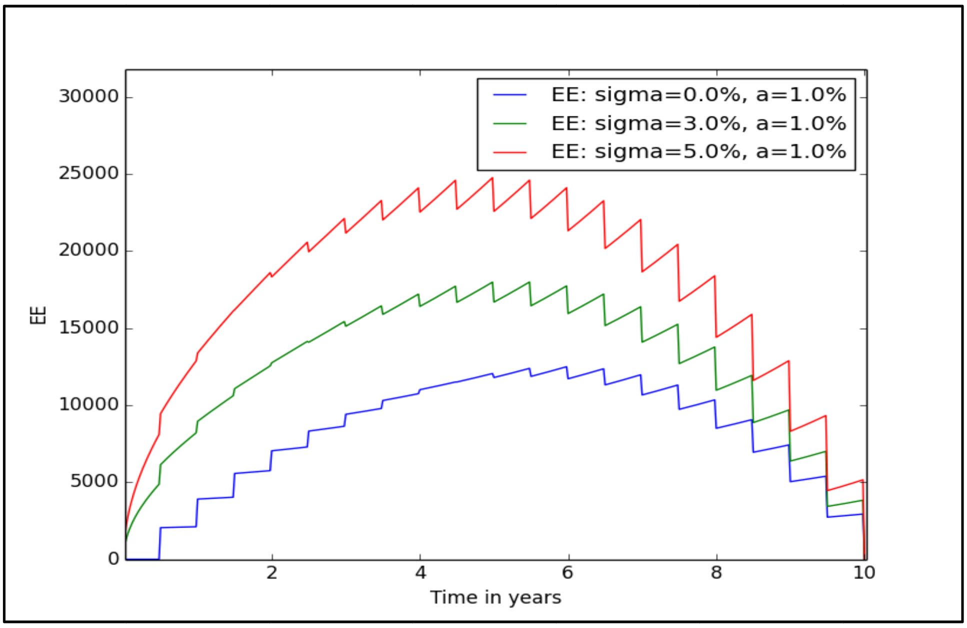


Figure 5-10: Exposure profiles for different values of sigma for a payer swap

a	Sigma (σ)	CVA	DV01
1%	0%	5 391.84	24.46
1%	3%	8 691.25	12.76
1%	5%	12 154.41	9.11

Table 5-10: Numerical results for different values of sigma for a payer swap under Hull-White model

5.2.1.2 Receiver swap

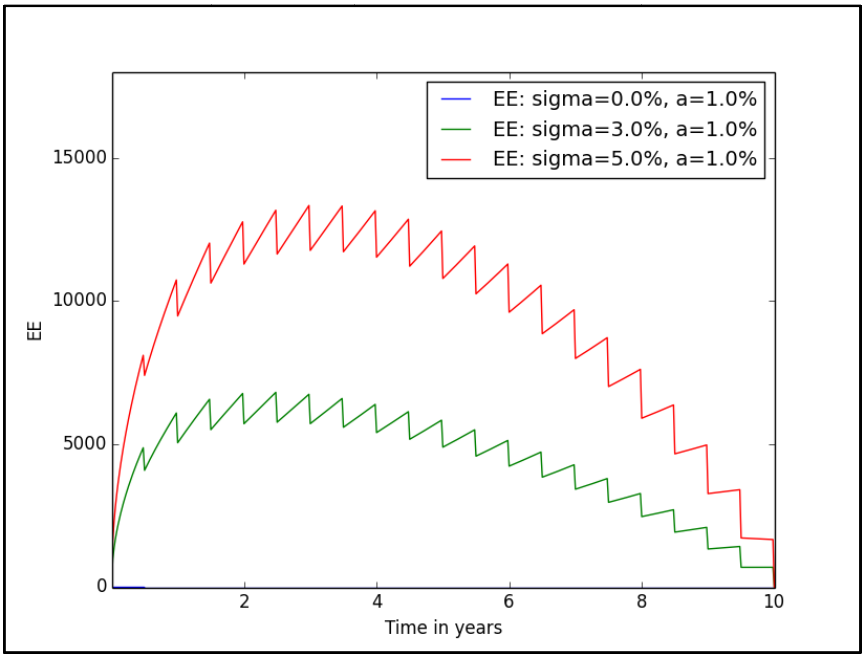


Figure 5-11: exposure profiles for a receiver swap under different values for sigma

a	Sigma (σ)	CVA	DV01
1%	0%	0	-1.56
1%	3%	3 271.64	-13.24
1%	5%	6 685.38	-16.86

Table 5-11: Numerical results for different values of sigma for a receiver swap under Hull-White model

5.2.2 The effect of the mean reversion parameter (a)

The mean reversion parameter has an inverse relationship with volatility. Other than that, it has the same effect on exposure profiles as the volatility parameter. Even the phenomenon that the absolute value of CVA DV01 decreases as CVA increases in the payer swap case, and increases in the receiver swap case, appears here. This is observed for the payer and receiver swaps.

5.2.2.1 Payer swap

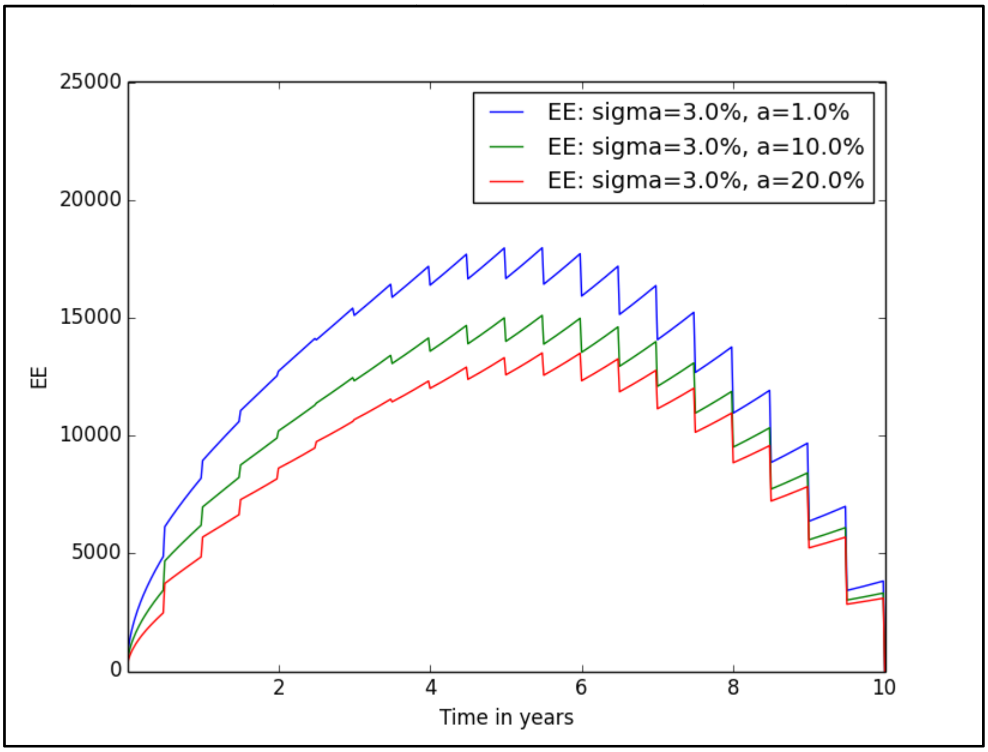


Figure 5-12: Exposure profiles for different values of a, under a payer swap

a	Sigma (σ)	CVA	DV01
1%	3%	8 691.25	12.76
10%	3%	7 165.75	15.27
20%	3%	6 278.96	17.63

Table 5-12: Numerical results for different values of a for a payer swap under Hull-White model

5.2.2.2 Receiver swap

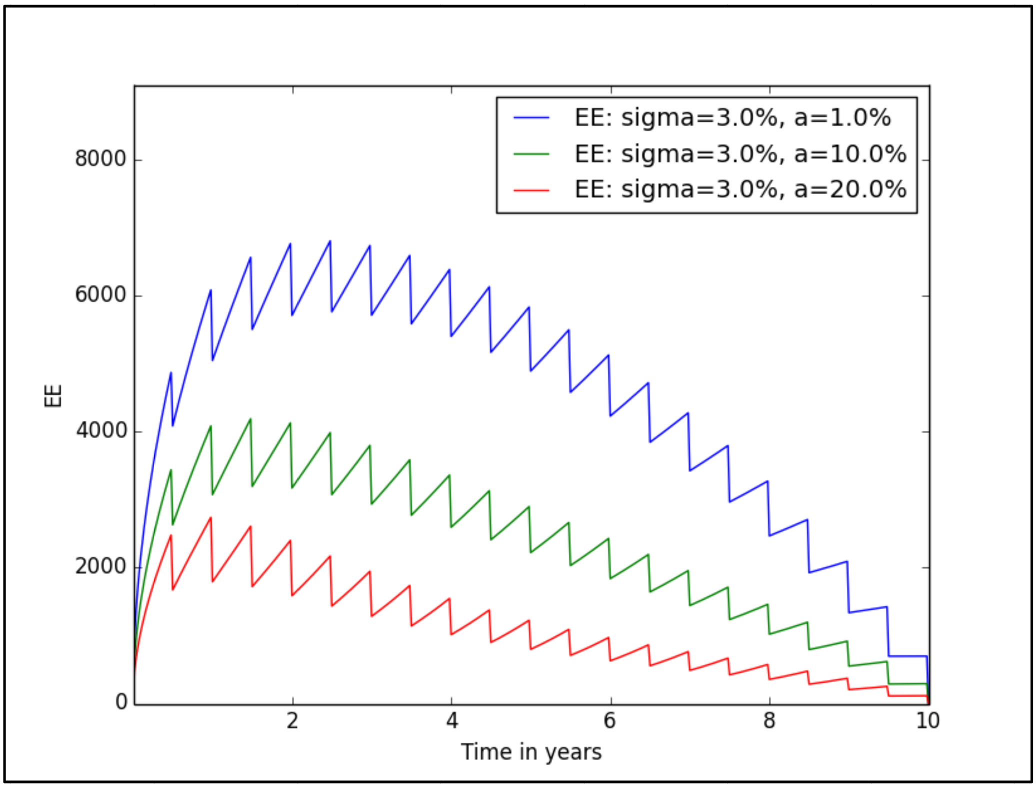


Figure 5-13: Exposure profiles for different values of a, under a receiver swap

a	Sigma (σ)	CVA	DV01
1%	3%	3 271.64	-13.24
10%	3%	1 757.00	-10.74
20%	3%	8 76.09	-8.38

Table 5-13: Numerical results for different values of a for a receiver swap under Hull-White model

5.2.3 Collateral agreements

This section will detail the calculated results of exposures of collateralised swaps under the Hull-White model. A collateral threshold of R1 000 was used, as in the Black model case.

5.2.3.1 Effect of volatility parameter

Payer swap

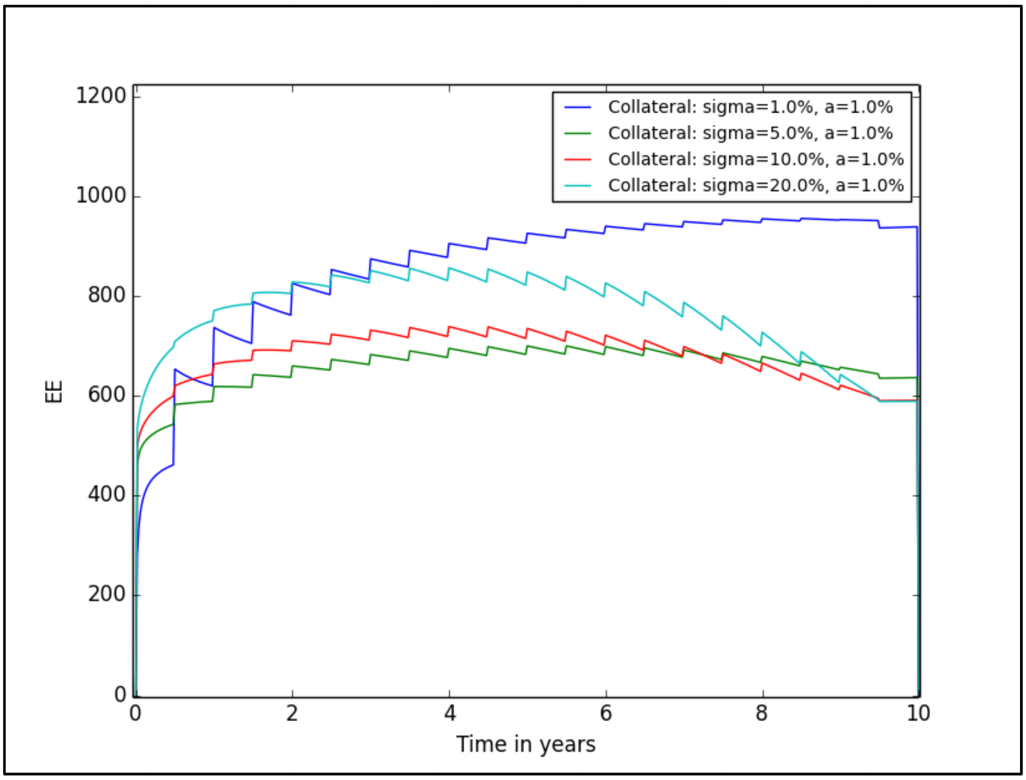


Figure 5-14: Exposure profiles for different values of sigma, under a collateralised payer swap

The EE profiles above show some very interesting shape changes for changes in sigma if the underlying swap is under a collateral agreement. One can clearly see how the profile falls at first to the green profile, then climbs again in exposure for the first few years. Exposures at the end of the lifetime of the swap contract fell, as expected following the explanation given in section 5.1.3.

CVA decreased in the payer swap case from the lowest to the intermediate volatility, as expected. From intermediate to the highest volatility, though, it increased again. This could indicate that under collateral agreements, there is an "optimal point" for volatility, neither too high nor too low, where CVA is at its lowest, keeping all other factors constant. This was not observed in Black's model. DV01 decreased for every increase in volatility, as observed in Black's model.

a	Sigma (σ)	CVA	DV01
1%	1%	570.23	1.40
1%	5%	447.50	0.39
1%	10%	466.78	0.19
1%	20%	530.96	0.08

Table 5-14: Numerical results for the various values of sigma for a payer swap under a collateral agreement

Receiver swap

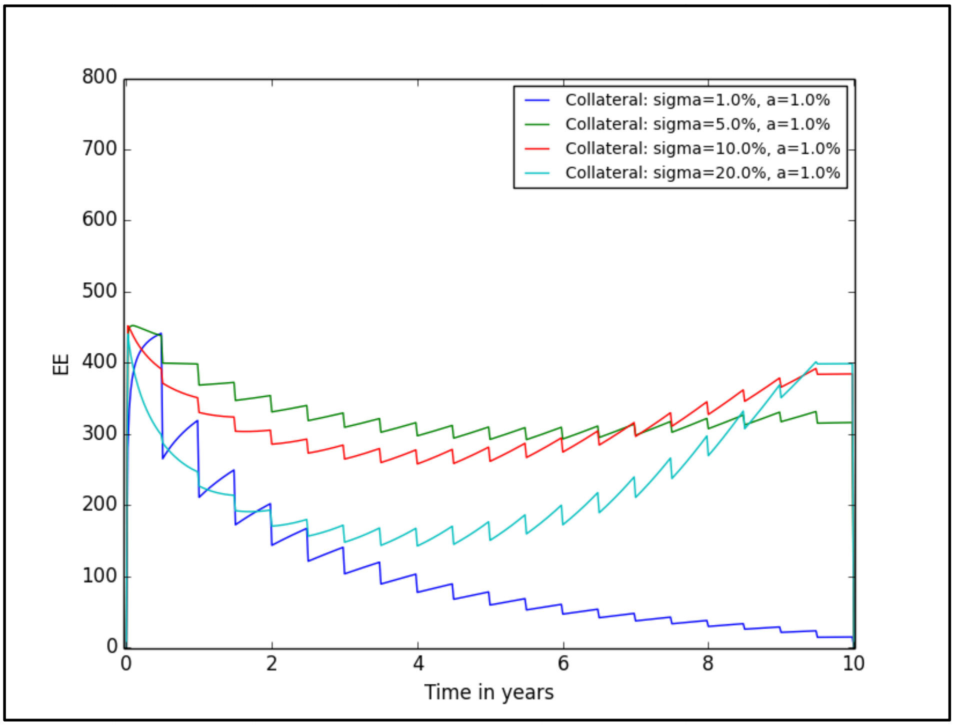


Figure 5-15: Exposure profiles for different values of sigma under a collateralised receiver swap

In the receiver swap case, increasing sigma up to high values seemed to create exactly the opposite scenario from a payer swap: an "optimal" value for sigma where the CVA is at its highest value. Refer to section 5.1.3 for an explanation of why lower sigma values pull exposure down in the case of a receiver swap.

a	Sigma (σ)	CVA	DV01
1%	1%	89.67	-1.29
1%	5%	228.69	-0.38
1%	10%	213.93	-0.19
1%	20%	152.18	-0.08

Table 5-15: Numerical results for the various values of sigma for a receiver swap under a collateral agreement

5.2.3.2 Effect of mean reversion parameter

Payer swap

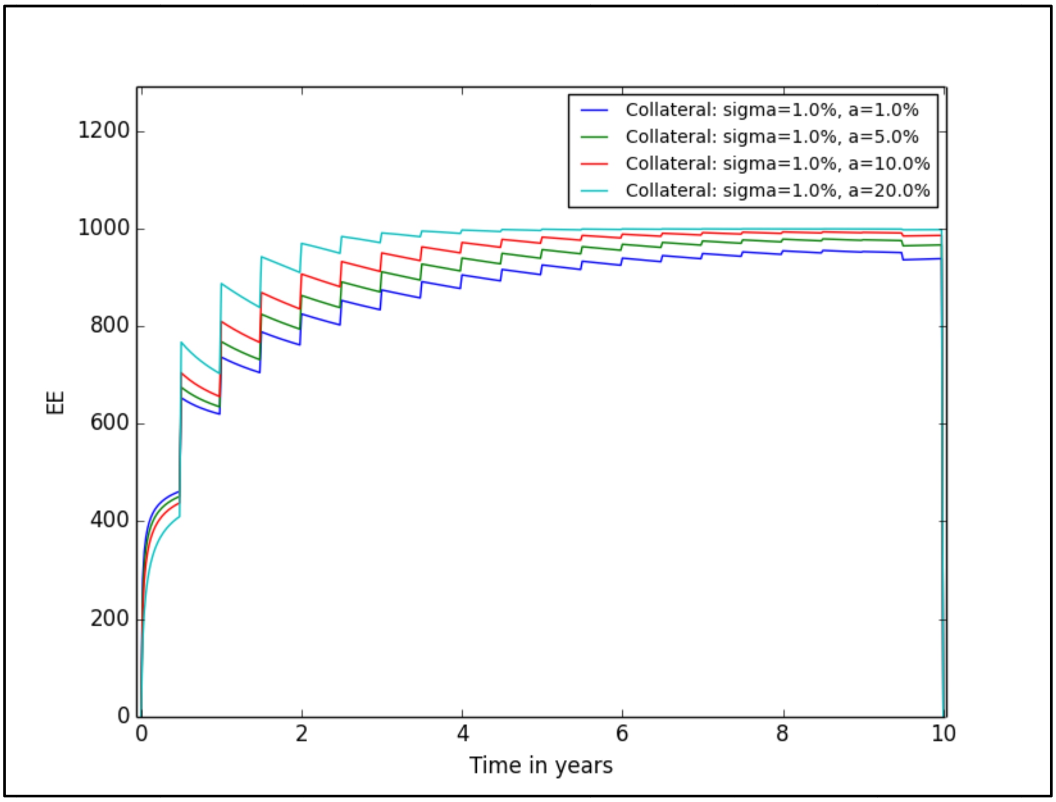


Figure 5-16: Exposure profiles for different values of a, under a collateralised payer swap

For every increase in a, EE curves shifted upwards, without any significant change in the shape of any of the curves. This is possibly due to a's effect of reducing overall volatility in the model, since interest rates stay closer to the mean, so interest rates are more likely to stay around the mean. At very low values for sigma, EE profiles tend to go upward towards the collateral threshold as illustrated by Figure 5-16. Since sigma was already low in this case, the increasing values for a probably just exacerbated this effect. It is unknown what changing levels of a would do at more levels of sigma, since that is not the main focus of this study. Studying the interaction effects between a and sigma requires a more in-depth study, and could be a topic for future research.

CVA values increased in accordance with the EE profiles, while DV01 also increased as a increased. This is understandable, since increasing a causes interest rates to stay around the mean more.

a	Sigma (σ)	CVA	DV01
1%	1%	570.23	1.40
5%	1%	588.95	1.49
10%	1%	607.02	1.57
20%	1%	627.80	1.68

Table 5-16: Numerical results for the various values of a for a payer swap under a collateral agreement

Receiver swap

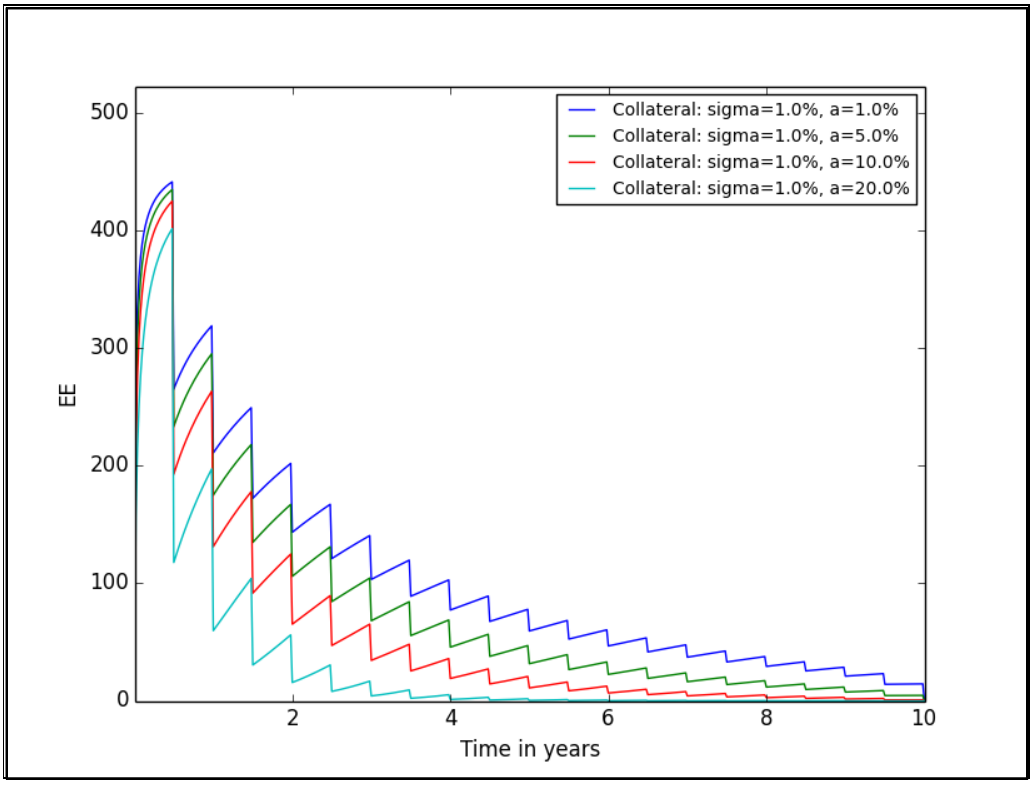


Figure 5-17: Exposure profiles for different values of a under a collateralised receiver swap

Increasing a in the receiver swap case lowered the value of CVA monotonically, while increasing interest rate risk, implied by the increasing absolute value of DV01.

a	Sigma (σ)	CVA	DV01
1%	1%	89.67	-1.29
5%	1%	70.87	-1.35
10%	1%	52.93	-1.39
20%	1%	31.71	-1.43

Table 5-17: Numerical results for the various values of sigma for a receiver swap under a collateral agreement

5.2.3.3 Overall effect of collateral

Naturally, collateral agreements reduced CVA significantly under the Hull-White model, as in Black's model. This is shown by

Figure 5-18 below.

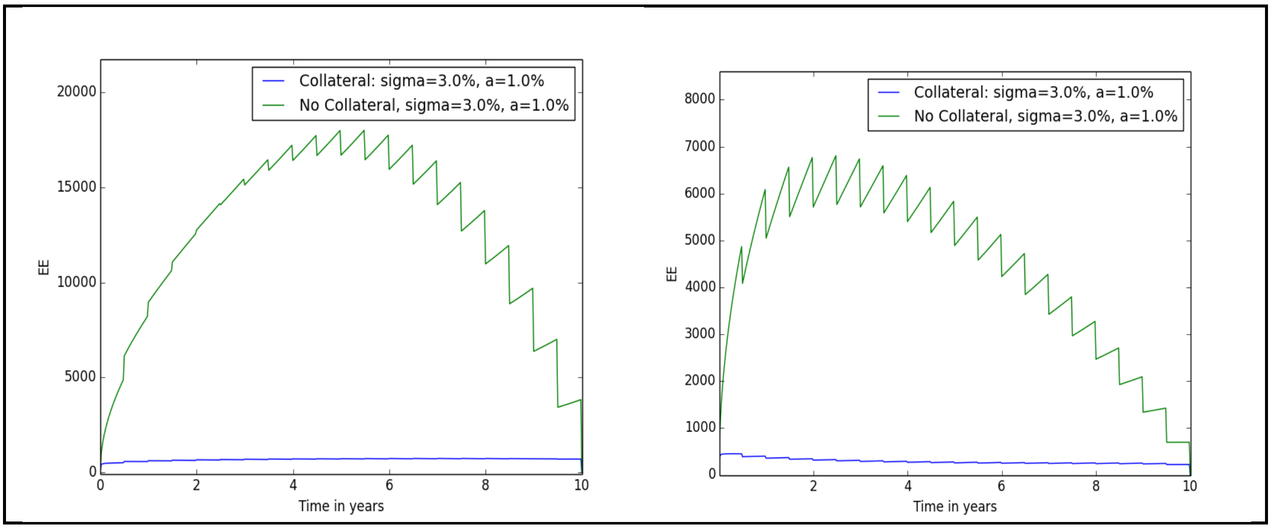


Figure 5-18: Left: Effects of collateral for a payer swap.
Right: Effects of collateral for a receiver swap

5.2.4 Netted Portfolio

The table below shows the portfolio of swaps that was analysed for credit risk. All had the same swap rate (the same scenario shown in section 2.1.2.1).

	Start Date	End Date	Principal
Swap 1	11/10/2014	11/10/2024	100 000
Swap 2	11/10/2017	11/10/2024	100 000
Swap 3	11/10/2020	11/10/2024	200 000
Swap 4	11/10/2021	11/10/2024	100 000

Table 5-18: Composition of the portfolio of swaps. All swaps were either receiver or payer swaps.

Figure 5-19 and Table 5-19, below, show the EE profiles and numerical results for portfolios of payer and receiver swaps.

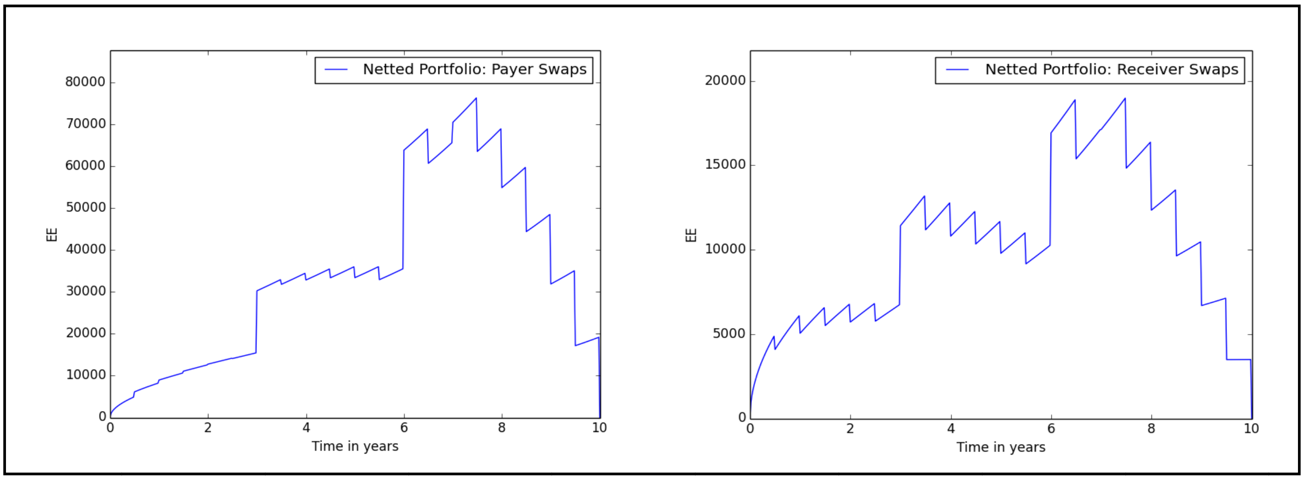


Figure 5-19 Left: exposure under a simple netted portfolio for a payer swap. Right: Exposure under a simple netted portfolio for a receiver swap.

	CVA	**CVA DV01**	**MtM**
Payer Swaps	20 797.94	19.84	28 118.01
Receiver Swaps	6 523.89	-23.15	-28 118.01

Table 5-19: Numerical results for the netted portfolios of payer and receiver swaps

Since these extremely simplified netted portfolios have their exposures calculated by simply adding together the individual exposures, the netted portfolio in the Hull-White case is essentially the same as under Black's model. A different netted portfolio, with different starting dates but the same end dates was used in this case for some variety. This produced the effect of a positive market value for party A (holding the payer swap) and the opposite effect for party B (holding the receiver swap).

5.3 Applied to South African data

This section contains results from applying the swaption model to actual, observed South African data. The table below lists observed swap rates for each of the dates that CVA was calculated on for a single swap. As always, probability of default is 10%, notional is R100 000, the swap ends on 20/02/2024 and payments are semi-annual. Since the swap was entered into on 20/02/2014, the swap rate was 8.66% throughout. The implied volatility matrices and term structures used on each date are given in Appendix B.

Date	Swap Rate
20/02/2014	8.66%
21/02/2014	8.67%
27/02/2014	8.59%
26/03/2014	8.40%
19/06/2014	8.18%

Table 5-20: Observed South African swap rates on relevant dates

The graph below is the profile as calculated on 20/02/2014, and shows the effectiveness of the calibration algorithm. CVA under Black's model was 3035.0613, while DV01 was 14.9571. Under the Hull-White model, CVA was 2960.1026, while DV01 was 13.0866. Current market value was zero. The fact that the graphs are nearly identical affirms the effectiveness of the calibration algorithm, since swaption prices calculated under Black's model from the implied volatility matrix are generally observed market prices.

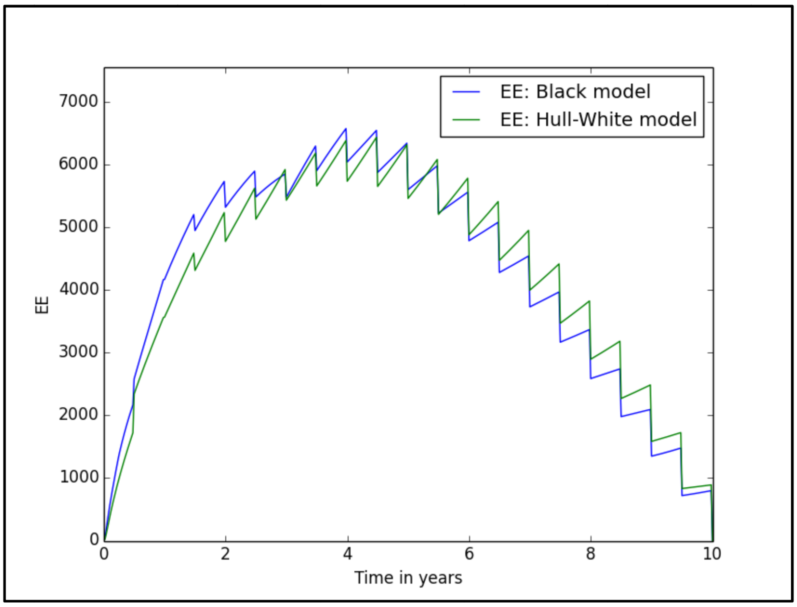


Figure 5-20: Comparison of the Hull-White and Black's model EE profiles.

Results for the Black and Hull-White models, applied to a South African interest rate environment, are given next. In the first case, no collateral agreements were assumed. Since the shape of the exposure profiles in both cases did not change drastically, all 5 profiles were plotted on the same set of axes in each case.

5.3.1 No collateral

CVA will here be calculated where no collateral agreement exists. This is done for both models, and a comparison between the two models will be made at the end of this subsection.

5.3.1.1 Black's model results

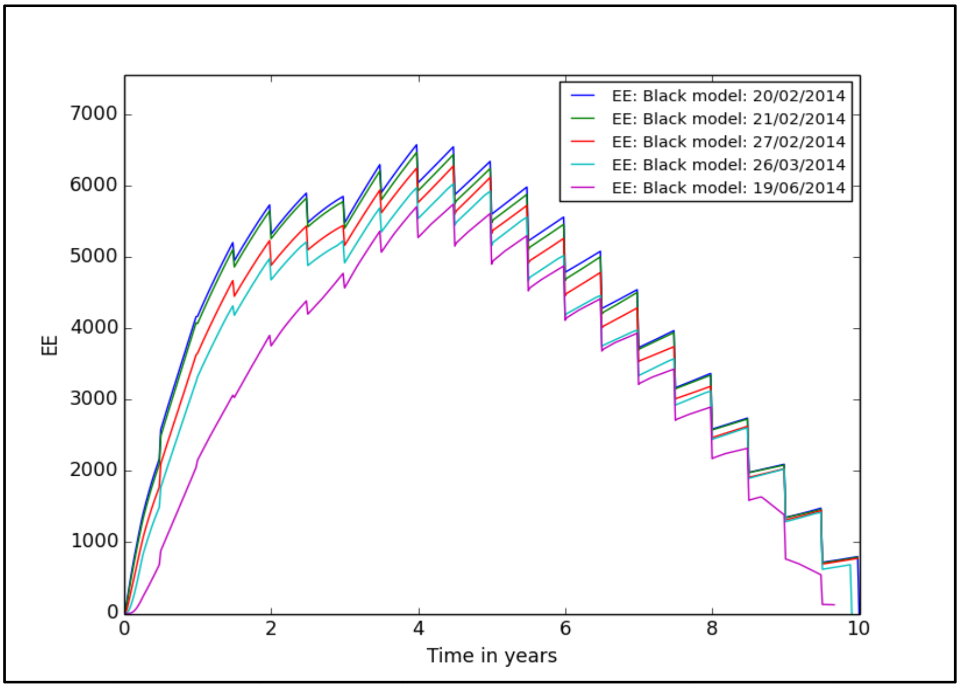


Figure 5-21: EE profiles calculated for each date for the same swap under Black's model

The graph above shows the exposure profile calculated on each respective date. The earliest date has the highest profile, and as time passes, the profiles get lower. Numerical results for each of the dates are shown in the table below. It can be seen that, while declining, exposure profiles lie very close to each other. The decline can be explained by a combination of: gradually lowering market swap rates, the shortening life span of the swap, and swaption implied volatility matrices that did not seem to increase or decrease significantly in volatility over the period that the dates fall in, that CVA was calculated for.

CVA values in Table 5-21 below reflect the lowering of the exposure profiles. Since the term structure is mostly upward sloping, and we are analysing a payer swap, the fact that DV01 fell in conjunction with CVA suggests that the changes in CVA was not due to volatility. As noted earlier, this is because DV01 and CVA tend to move in opposite directions with changes in volatility, under similar circumstances, under Black's model (see section 5.1.1). Market values (in the MtM column) show gradually decreasing negative values over the course of the dates listed, implying that the counterparty carries the credit risk (assuming both parties had the same uniform probability of default distribution).

Date	CVA	CVA DV01	MtM
20/02/2014	3 035.06	14.96	0.00
21/02/2014	2 988.63	14.78	67.57
27/02/2014	2 841.78	14.25	-475.50
26/03/2014	2 711.26	13.56	-1770.24
19/06/2014	2 378.64	12.17	-3247.10

Table 5-21: Numerical results for Black's model under no collateral agreements

CVA values for the counterparty (receiver swap) are shown in Table 5-22 below. CVA values for the counterparty is larger than CVA for the party holding the payer swap, which corresponds to market values on the last three dates. It is interesting to note that as market values indicate increasingly that the counterparty carries the most credit risk with passing time, CVA correspondingly increases for the counterparty and decreases for the party holding the payer swap. This widened the gap between CVA values further (see section 25 for the relationship between CVA and counterparty CVA).

Date	CVA	CVA DV01	MtM
20/02/2014	3 186.79	-17.05	0.00
21/02/2014	3 275.64	-17.36	-67.57
27/02/2014	3 474.13	-18.26	475.50
26/03/2014	3 744.61	-19.08	1 770.24
19/06/2014	3 746.41	-19.86	3 247.10

Table 5-22: Numerical results for counterparty CVA under Black's model

5.3.1.2 *Hull-White model results*

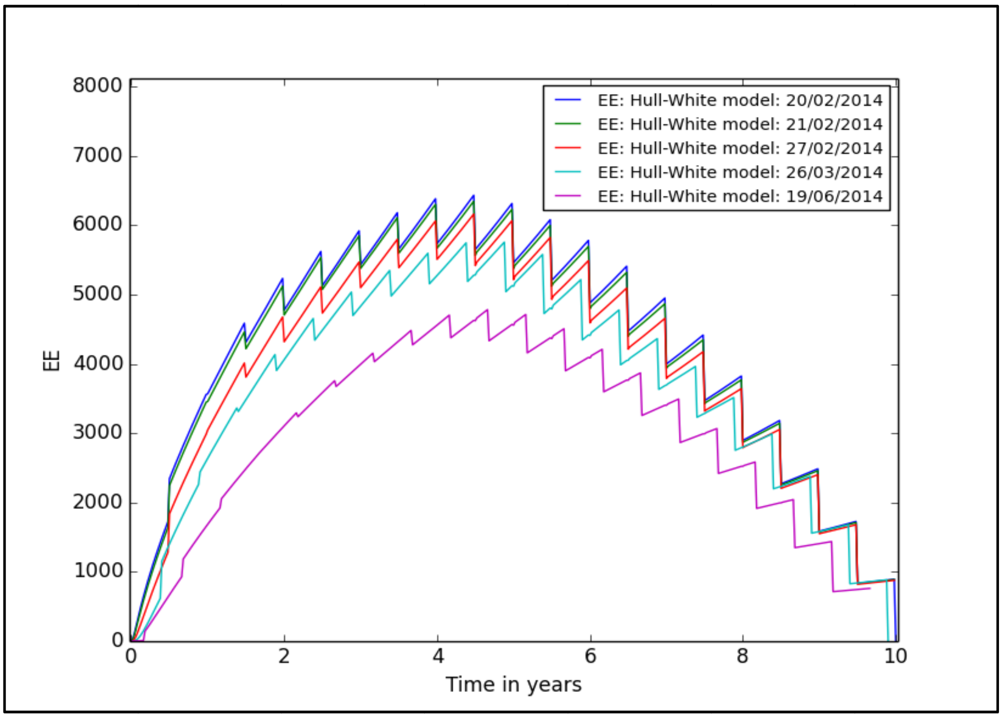


Figure 5-22: EE profiles calculated for each date for the same swap under the Hull-White model

Exposure profiles under the Hull-White model under each respective date are shown above. Profiles for the first four dates are quite close, while the profile for the fifth date is markedly lower than the others, as expected.

Table 5-23 shows the numerical results derived from the EE profiles just discussed. The benefit of the Hull-White model here is that we can see what the model parameters were for each curve. Unfortunately, the parameters themselves do not say much: sigma and a, which have opposite

Date	**CVA**	**CVA DV01**	**MtM**	**Calibrated model parameters**	
				a	**sigma**
20/02/2014	2 960.10	13.09	0.00	10.61%	2.29%
21/02/2014	2 914.87	12.88	67.57	9.15%	2.17%
27/02/2014	2 762.34	12.29	-475.50	8.33%	2.08%
26/03/2014	2 562.85	11.46	-1770.24	5.60%	1.85%
19/06/2014	2 050.36	10.42	-3247.10	-2.31%	1.17%

Table 5-23: Numerical results for Hull-White model under no collateral agreements

effects, both declined at the same time, making it difficult to make any kind of statement about an overall volatility trend. It could be argued that a declined more drastically than sigma, even becoming negative, but sigma was also much smaller than a to start with on 20/02/2014.

Table 5-23 also shows that CVA values decline gradually with sharper decreases from dates 21/02/2014 to 27/02/2014 (considering the short time) and from dates 26/03/2014 to 19/06/2014.

In the first case, the swap rate dropped by 8 basis points in less than a week, while the mean reversion rate also fell, which presumably was not offset in full by the small decline in sigma at the same time. The second decrease can be attributed to the large time gap (about 11 to 12 weeks), as well as the less steep decline in market swap rates than in the first case.

CVA values for the counterparty are shown in Table 5-24 below. Clearly, CVA values for the counterparty are larger than CVA for the party holding the payer swap. This is in accordance with market values for the last three dates, which implies that the counterparty carries the credit risk. While under Black's model CVA for the counterparty increased monotonically as time went on, CVA for the counterparty under the Hull-White model decreased again on the last date. This could be due to the Hull-White model's more complex parameterisation and its time independent parameters.

Date	CVA	CVA DV01	MtM	Calibrated model parameters	
				a	sigma
20/02/2014	3 102.16	-18.85	0.00	10.61%	2.29%
21/02/2014	3 195.54	-19.21	-67.57	9.15%	2.17%
27/02/2014	3 399.56	-20.27	475.50	8.33%	2.08%
26/03/2014	3 614.37	-21.10	1770.24	5.60%	1.85%
19/06/2014	3 472.04	-21.59	3247.10	-2.31%	1.17%

Table 5-24: Numerical results for counterparty CVA with no collateral under Hull-White model

5.3.1.3 Comparison of the two models

The shapes of the two models' EE profiles are slightly different. This could be explained by the fact that Black's model was used with time-varying volatilities, while under the Hull-White model, parameters were constant for a given profile. Another apparent difference is that the EE profile under the Hull-White model for the last date is even more significantly lower than the other profiles, than under Black's model. Once again, this could be explained by the constant parameters of the Hull-White model, as opposed to time-varying volatility under Black's model. Another explanation for both of those differences though, is the fact that the Hull-White model has one more parameter than Black's model.

CVA values from Table 5-23 and Table 5-21 show that CVA values for the two models can differ quite substantially in some cases. These differences become especially large for the last two dates. Since the CVA is calculated from the EE profile, the explanation for these differences is the same as above.

5.3.2 Collateral

In this section, credit risk on the swap that was considered in the previous section will be considered again under a collateral agreement with a threshold amount of R1 000. Market swap values were calculated under simplifying assumptions given in the research methodology, under section 4.6 .

5.3.2.1 Black's model results

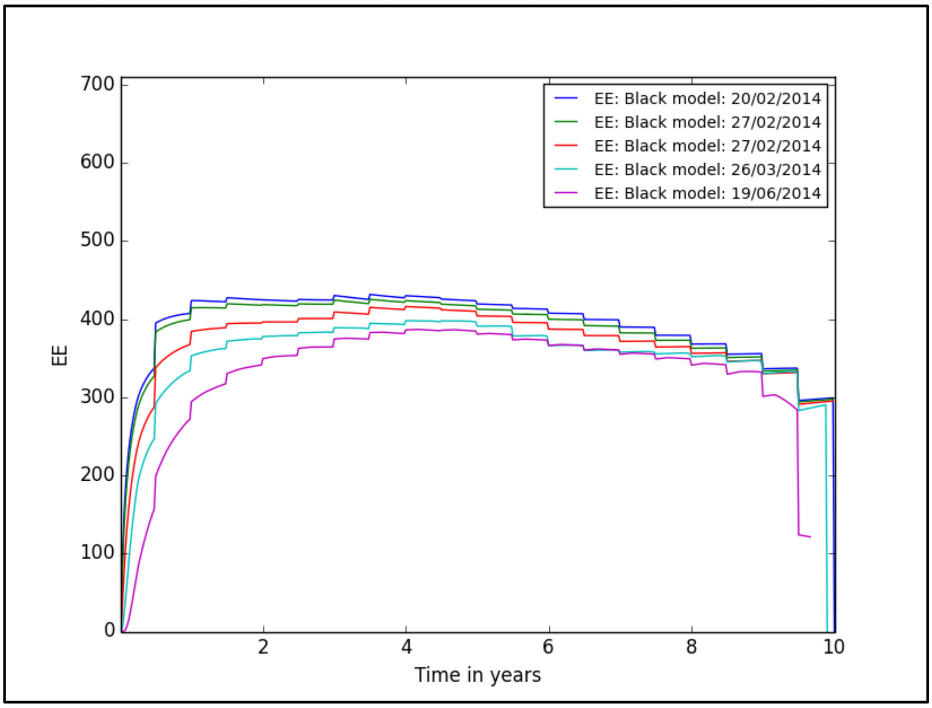


Figure 5-23: EE profile under a collateral agreement for each date under Black's model

EE profiles calculated under Black's model are shown above. As in the previous section, the earliest date has the highest profile, and as time progresses, the profiles gradually start shifting downwards.

What is readily apparent in Figure 5-23 is that the profiles toward the end of the swap's lifetime. The most likely explanation for this is that the swap at any date will show lower exposure close to the end of its lifetime, since there are less cash flows expected then. This is seen even in EE profiles where no collateral agreements are present, but it is less apparent. By contrast, larger exposure is expected at the beginning of a swap's lifetime, because even a small difference in market conditions will affect a larger number of cash flows. At all times, the swap is therefore affected by the same factor in this instance. Since market conditions did not change drastically from 20/02/2014 to 19/06/2014, the swap shows similar exposures toward the end of its lifetime on each of the dates.

Date	CVA	CVA DV01	MtM
20/02/2014	271.87	1.01	0.00
21/02/2014	267.24	1.00	67.57
27/02/2014	257.08	0.98	-475.50
26/03/2014	244.26	0.92	-1000.00
19/06/2014	224.61	0.91	-1000.00

Table 5-25: Numerical results for CVA obtained under Black's model for a collateralised swap contract

Numerical results are shown in Table 5-25 above. There was no significant trend in implied volatility over the time of those dates listed above. The decline in CVA can therefore best be explained by the lowering market swap rates and the decline in remaining life of the swap as time progresses.

In Table 5-26 below, CVA values for the counterparty are shown. Once again, counterparty CVA exceeds CVA of the party we are analysing, corresponding with market values indicating that the counterparty carries the larger risk.

Date	CVA	CVA DV01	MtM
20/02/2014	387.36	-1.36	0.00
21/02/2014	392.73	-1.35	-67.57
27/02/2014	407.46	-1.32	475.50
26/03/2014	421.95	-1.26	1000.00
19/06/2014	432.21	-1.25	1000.00

Table 5-26: Numerical results for counterparty CVA for collateralised swap under Black's model

5.3.2.2 Hull-White model results

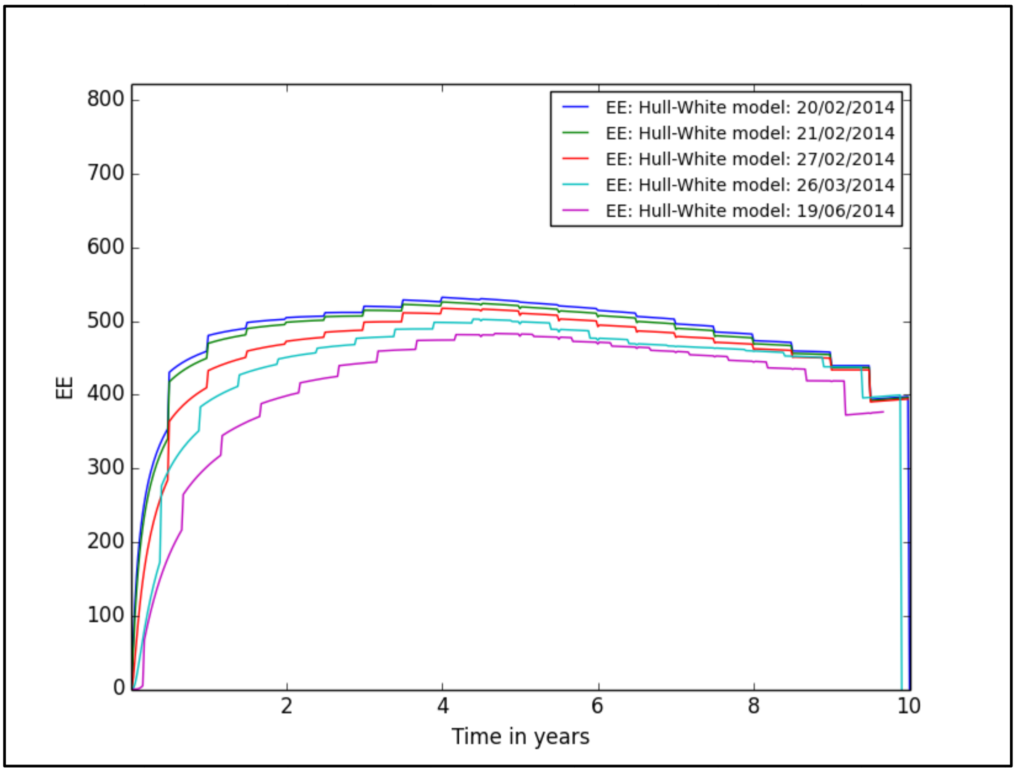


Figure 5-24: EE profile for a payer swap under a collateral agreement for each date under the Hull-White model

EE plots under the Hull-White model are shown in Figure 5-24 above. Once again, we notice the profiles converging at the end of the life of the swap contract, for the same reasons as mentioned in the previous section.

What is interesting to note here, is the DV01 values for CVA. It falls gradually, except that on the last date where it increases again.

Date	CVA	CVA DV01	MtM	Calibrated model parameters (same as above)	
				a	sigma
20/02/2014	329.84	1.21	0.00	10.61%	2.29%
21/02/2014	325.25	1.19	67.57	9.15%	2.17%
27/02/2014	313.61	1.16	-475.50	8.33%	2.08%
26/03/2014	299.67	1.09	-1000.00	5.60%	1.85%
19/06/2014	275.72	1.15	-1000.00	-2.31%	1.17%

Table 5-27: Numerical results for CVA obtained under the Hull-White model for a collateralised swap contract

Date	CVA	CVA DV01	MtM	Calibrated model parameters	
				a	sigma
20/02/2014	327.52	-1.27	0.00	10.61%	2.29%
21/02/2014	329.28	-1.26	-67.57	9.15%	2.17%
27/02/2014	345.41	-1.22	475.50	8.33%	2.08%
26/03/2014	360.76	-1.15	1000.00	5.60%	1.85%
19/06/2014	378.56	-1.21	1000.00	-2.31%	1.17%

Table 5-28: Numerical results for counterparty CVA with collateral under Hull-White model

CVA values for the counterparty are shown in Table 5-28 above. Counterparty CVA increased monotonically under collateralisation, unlike the case for no collateral under Hull-White, but was still in accordance with market values for the swap contract.

5.3.2.3 Comparison of the two models

The overall shape of the EE profiles under the two models was very similar. However, this could be misleading because of the way the profiles converge in both cases.

Table 5-25 and Table 5-27 show even larger differences in CVA for the two models under a collateral agreement. These differences are seen on all the dates, and are especially large when considering the percentage of the values that the differences are, e.g. the smallest difference is over 10% of CVA value, while the largest is well over 20% of CVA value. These differences could be attributed to the Hull-White model having an extra parameter for describing interest rate evolution and time independent parameters while the Black's model parameter varies with time.

6 Conclusion

Since there were quite a few research objectives in this paper, conclusions on each will be made in a case-by-case fashion here. Possible areas for future research will be listed afterwards.

Firstly, the comparison of the swaption approach versus the simplistic mark-to-market approach showed that the mark-to-market approach did in most cases indicate which party carried the most credit risk. However, the simplistic approach gives us far less information on the credit risk involved between the two parties. One of the major shortcomings of the simplistic approach is that the risk cannot be divided to see how it is distributed across the two parties involved in the contract. Since CVA can be calculated for each party, the swaption approach provides a means to obtain this distribution of risk. Another shortcoming of the simplistic approach is that no exposure profile can be calculated from it. Exposure profiles provide an excellent means to ascertain what the average risk will be for each date in the future, and the swaption approach readily provides the means to obtain such profiles.

The DV01 measure was relatively easy to calculate in each case. Since this measure effectively gave an approximation to the exposure of CVA itself to interest rate risk, it could allow parties to hedge against any adverse changes in CVA.

It was very clear that having a collateral agreement on the swap contract reduced credit risk considerably. The extent of this effect depends naturally on the threshold, since it was shown that exposure under collateral agreements never climb above the threshold amount.

Because of time constraints and the need to limit the size of this paper, netting could only be considered in very simple and almost unrealistic cases. The results from netting were not very revealing. All that could be concluded from netting results was that large drops and jumps in exposure could be expected when a swap matures and when a swap is entered into, respectively. This does not include offsetting swaps.

Model parameters were shown to have a very large impact on CVA and EE profiles. Since model parameters are a reflection of market conditions regarding interest rates and swap rates, understanding the effects of model parameters on CVA will allow parties to take appropriate action when certain market conditions are expected in the near future.

A comparison of CVA values under the two models revealed some significant differences, especially when tested under collateral agreements. This could provide motivation for using the Hull-White model instead of Black’s model, since academic literature explicitly recommends term structure models for CVA calculation when using the semi-analytical approach. However, since Black’s model had a time dependent parameter, while the Hull-White model’s parameters were time independent, this may be slightly misleading.

Exposures and CVA calculated under Black’s model and the Hull-White model showed similar patterns when calculated on SA data, but the levels of CVA often differed slightly. This could be attributed to the fact that time-varying volatility was used under Black’s model, calculated from the implied volatility matrix on each date, as opposed to constant parameters under Hull-White. It could also be attributed to the fact that the Hull-White model has an extra parameter with which it can characterise interest rates. Because the Hull-White model provides information on how interest rates evolve through time, while Black's model does not, it could be that the Hull-White model takes account of something that Black's model does not. While investigating the effect of volatility on CVA under collateral agreements, CVA had certain "optimal" values under the Hull-White model. This was not observed under Black's model, even when testing volatility for a wide range of values.

The calibration method used was a very thorough one, although costly in terms of processing power required. It was taken in favour of simpler calibration methods so that the amount that differences in models could be attributed to inefficient calibration could be minimised.

6.1 Areas earmarked for future research

There were plenty of questions left unanswered, due to limitations regarding time and the size of this paper. Those questions are listed here.

In this paper, the assumptions made for netting agreements were very limiting. Future research areas could be to investigate possible methods for calculating CVA under more complex netting agreements, such as allowing for different swap rates in the portfolio.

Some simplifying assumptions were also made regarding collateral agreements, mainly disregarding different periods for the posted amount and the current swap rate. Other factors such as the grace period and independent amount were also disregarded. The inclusion of these factors requires that the formulas for CVA under collateral agreements get more complex, and is therefore a potential area for further research.

Swap rates were not marked to market regularly as time went on. As this is an available risk management technique for swaps, the analysis was not as complete as it could have been. Also, swap market values are calculated differently under collateral agreements. This was not done due to the same limitations mentioned, and so the comparison of CVA under collateral agreements with the mark-to-market method was not entirely accurate.

It was realised that the two parameters of the Hull-White model might be subject to interaction effects, e.g. the a parameter might have a different effect on CVA for different values of sigma. This requires a more rigorous and in-depth study than was done in this paper on these interaction effects alone.

It was mentioned in the conclusion that because the Hull-White model had time independent parameters while Black's model had a time dependent parameter, the comparison of the two models may have been slightly misleading. A future study where the Hull-White model has time-dependent parameters also, might provide a more accurate comparison of CVA calculation under the two models.

7 Appendices

Appendix A - Proofs

These proofs are adapted from Bateson (2011:306-312). All variables have the same definitions as described in the literature review (chapter 2), unless stated otherwise.

Hull-White One-Factor Model

The Hull-White One-Factor model describes the change in the short rate, $r(t)$, as

$$\boldsymbol{dr(t) = [\theta(t) - ar(t)]dt + \sigma dz(t)} \quad \textbf{(A.1)}$$

Where

- a is the mean reversion parameter
- σ is the volatility parameter

Using the method of integrating factors on **(A.1)**, we obtain

$$r(t) = r(0)e^{-at} + e^{-at}\int_0^t \theta(u)e^{au}du + \sigma e^{-at}\int_0^t e^{au}dz$$

From this, the expectation and variance of the spot rate are given by

$$E[r(t)] = r(0)e^{-at} + e^{-at}\int_0^t \theta(t)e^{au}du$$

$$V[r(t)] = \frac{\sigma^2}{2a}[1 - e^{-2at}]$$

From which an analytical form for $\theta(t)$ can be obtained

$$\theta(t) = \frac{\partial f(0,t)}{\partial t} + af(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at})$$

which can be approximated by

$$\frac{\partial f(0,t)}{\partial t} + af(0,t)$$

Substituting back into **(A.1)**, we can approximate the spot rate drift as

$$\frac{\partial f(0,t)}{\partial t} + a[f(0,t) - r]$$

which shows us that mean reversion gets bigger, or stronger, as the difference between the spot and forward rate increases.

In a similar fashion to the Black-Scholes model, a PDE for the Hull-White model can also be obtained. It is given by

$$\frac{\partial V(r,t)}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V(r,t)}{\partial r^2} + (\theta(t) - ar)\frac{\partial V(r,t)}{\partial r} - rV(r,t) = 0 \qquad \textbf{(A.2)}$$

The price of a discount bond as seen at time r, at time t, that matures at time T with a payoff of one unit of currency, can be obtained from **(A.2)** and is given by

$$D(r,t,T) = A(t,T)e^{-B(t,T)r(t)} \qquad \textbf{(A.3)}$$

where

- $B(t,T) = \frac{1-e^{-a(T-t)}}{a}$
- $lnA(t,T) = \ln\left(\frac{D(0,T)}{D(0,t)}\right) + B(t,T)f(0,t) - \frac{1}{4a^3}\sigma^2(e^{-aT} - e^{-at})(e^{2at} - 1)$

Since the spot rate $r(t)$ follows a normal process, we can deduce from **(A.3)** that the discount bond price $D(r,t,T)$ follows a log normal process. The following useful result was used for obtaining closed form solutions for the various versions of swaptions under the Hull-White one-factor model in this paper

$$D(t,s,T) = \frac{D(0,t,T)}{D(0,t,s)} e^{-\frac{1}{2}v^2(t,s,T) - v(t,s,T)Z} \qquad \textbf{(A.4)}$$

$v(t,T,s)$, the volatility of the discount bond, is given by

$$v^2(t,s,T) = \sigma^2 \left(\frac{1-e^{a(T-s)}}{a}\right)\left[\frac{1-e^{-2a(s-t)}}{2a}\right] \qquad \textbf{(A.5)}$$

Proof of the swaption pricing formula under the Hull-White model:

In the following sections, assume that all swaps and swaptions have the same characteristics as those in sections 2.3 and 2.4

Some Preliminaries

In Bateson (2011:23) the forward rate as seen at time t, between times T and T + h, is defined as

$$f(t,T,T+h) = \frac{1}{h}\left(\frac{D(t,T)}{D(t,T+h)} - 1\right) \qquad \textbf{(A.6)}$$

Bateson (2011:26-27) defines a floating rate note as an instrument that pays a floating rate coupon f at the end of each period of length h. The present value of the floating rate note is given by the present value of all future floating rate payments plus the principal at maturity

$$\boldsymbol{P_{FRN}(0,T) = \sum_{i}^{N} D(0,t_i + h) f_i\, h + D(0,T)} \quad \textbf{(A.7)}$$

By substituting **(A.6)** into the above equation, we can show that **(A.7)** is always equal to 1, as shown below

$$\begin{aligned}\boldsymbol{P_{FRN}(0,T)} &= \boldsymbol{\sum_{i}^{N} D(0,t_i + h) f_i\, h + D(0,T)} \\ &= \boldsymbol{\sum_{i}^{N} D(0,t_i + h) \frac{1}{h}\left(\frac{D(0,t_i)}{D(0,t_i + h)} - 1\right) h + D(0,T)} \\ &= \boldsymbol{\sum_{i}^{N} \big(D(0,t_i) - D(0,t_i + h)\big) + D(0,T) = D(0,t_0) = D(0,0) = 1}\end{aligned}$$

Bateson also defines the value of an interest rate swap as the difference between the present value of the difference between the floating and fixed payments. In the case of a payer swap (pay fixed, receive floating), the value is defined as

$$S(0,0,T) = PV\ Floating\ Leg - PV\ Fixed\ Leg = \sum_{j}^{N} D\big(0,0,t_j + h_j\big) f_j h_j - \sum_{i}^{M} D(0,0,t_i) C l_i$$

where

- l_i is the length of the period in which the i-th coupon payment takes place
- h_i is the length of the period in which the i-th floating rate payment takes place

In this paper, all coupon periods and floating rate periods were assumed to have the same length period, so then $V_{swap}(0,T)$ simplifies to

$$\begin{aligned}\boldsymbol{V_{swap}(0,T)} &= \boldsymbol{PV\ Floating\ Leg - PV\ Fixed\ Leg} \\ &= \boldsymbol{\sum_{j}^{N} D\big(0,t_j + h\big) f_j h - \sum_{i}^{M} D(0,t_i) C.h}\end{aligned} \quad \textbf{(A.8)}$$

Upon entering into the swap, the swap rate, C, is set such that $V_{swap}(0,T) = 0$, therefore

$$\sum_{j}^{N} D\big(0,t_j + h\big) f_j h = \sum_{i}^{M} D(0,t_i) C.h$$

$$\rightarrow \quad C(0,0,T) = \frac{\sum_j^N D\big(0, t_j + h\big) f_j h}{\sum_i^M D(0,t_i) h}$$

Since the numerator of this equation for C is equal to the value for a floating rate note minus the last term, and it was already proven that a floating rate note's price upon entering into the contract is equal to 1 (assuming the principal is 1), the formula for C becomes

$$\boldsymbol{C(0,0,T) = \frac{1 - D(0,T)}{\sum_i^M D(0,t_i) h}}$$

which proves equation (5). Note that C gives the swap rate for an IRS starting at time 0 (the present) and matures at time T. For this reason, it will be more convenient to express equation (5) as

$$\boldsymbol{C(0,0,T) = \frac{1 - D(0,0,T)}{\sum_i^M D(0,0,t_i) h}}$$

or more generally,

$$\boldsymbol{C(t,s,T) = \frac{1 - D(t,s,T)}{\sum_i^M D(t,s,t_i) h}} \qquad \textbf{(A.9)}$$

which is the formula for the swap rate observed at time t, for an IRS starting at time s and maturing at time T.

Note that from here on, s_t will be an alternative notation for $C(t,T,T_M)$, the swap rate observed at time t, for a swap that starts at time T and ends at time T_M. $S_{t,K}$ will be an alternative notation for $S(t,T,T_M)_K$, the value of a swap with swap rate K as observed at time t

Swaption Formula Derivation:

The price of a swaption is generally given by formula (10)

$$P_{HW}(0,T,T_M) = \xi(0) E_0 \left[\frac{\max(S(T,T,T_M),0)}{\xi(T)}\right]$$

Since our chosen numeraire is the zero coupon discount bond, the current, time 0 price of a European swaption expiring at time T on an IRS maturing at time T_M, is derived from the above formula as

$$P_{HW}(0,T,T_M) = E_0\big[\boldsymbol{D(0,0,T)}\max\big(S_{t,K},0\big)\big]$$

$$\rightarrow P_{HW}(0,T,T_M) = \boldsymbol{L}E_0\left[\boldsymbol{D(0,0,T)}\max(\boldsymbol{RS(T,T,T_M) - K, 0})\sum_{t_i > T}^{T_M} \boldsymbol{D(T,t_i)h}\right]$$

Substituting for $R(T,T,T_M)$ from equation **(A.9)** we have

$$P_{HW}(0,T,T_M) = \boldsymbol{L}\,\boldsymbol{D}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{T})E_0\left[\max\left(\boldsymbol{1}-\boldsymbol{D}(\boldsymbol{0},\boldsymbol{T},\boldsymbol{T_M})-\boldsymbol{K}\sum_{t_i>T}^{T_M}\boldsymbol{D}(\boldsymbol{0},\boldsymbol{T},\boldsymbol{t_i})\boldsymbol{h}\,,\boldsymbol{0}\right)\right] \quad \textbf{(A.10)}$$

Since each of the discount bonds in the above equation is stochastic, each depending on a standard normal random variable, as shown in equation **(A.4)**, rewriting each discount bond as a function of a specific standard normal value, Z is convenient

$$\boldsymbol{D}(\boldsymbol{0},\boldsymbol{t},\boldsymbol{T}) = \frac{\boldsymbol{D}(\boldsymbol{0},\boldsymbol{T})}{\boldsymbol{D}(\boldsymbol{0},\boldsymbol{t})}\boldsymbol{e}^{-\frac{1}{2}\boldsymbol{v}(\boldsymbol{0},\boldsymbol{t},\boldsymbol{T})-\boldsymbol{v}(\boldsymbol{0},\boldsymbol{t},\boldsymbol{T})\boldsymbol{Z}} = \boldsymbol{\lambda}(\boldsymbol{t},\boldsymbol{T},\boldsymbol{Z})$$

From Equation **(A.10)**, it is clear that the option is exercised for

$$\boldsymbol{1}-\boldsymbol{D}(\boldsymbol{0},\boldsymbol{T},\boldsymbol{T_M}) > K\sum_{t_i>T}^{T_M}\boldsymbol{D}(\boldsymbol{0},\boldsymbol{T},\boldsymbol{t_i})\boldsymbol{h}$$

or alternatively, all values of $Z \sim N(0,1)$ such that

$$\boldsymbol{1}-\boldsymbol{\lambda}(\boldsymbol{T},\boldsymbol{T_M},\boldsymbol{Z}) > K\sum_{t_i>T}^{T_M}\boldsymbol{\lambda}(\boldsymbol{T},\boldsymbol{t_i},\boldsymbol{Z})\boldsymbol{h} \quad \textbf{(A.11)}$$

Therefore we have to find Z^* such that

$$\boldsymbol{1}-\boldsymbol{\lambda}(\boldsymbol{T},\boldsymbol{T_M},\boldsymbol{Z}^*) = \boldsymbol{K}\sum_{t_i>T}^{T_M}\boldsymbol{\lambda}(\boldsymbol{T},\boldsymbol{t_i},\boldsymbol{Z}^*)\boldsymbol{h}$$

From **(A.4)**, it is clear that $\lambda(T,Z)$ is a decreasing function of Z, therefore the left hand side of the equation above is an increasing function of Z^*, while the right and side is a decreasing function. It is clear then that **(A.11)** is satisfied for all values of $Z > Z^*$. Since the expectation in **(A.10)** is an integral, we integrate over all values of $Z > Z^*$, therefore we have from **(A.10)**

$$\boldsymbol{P_{HW}}(\boldsymbol{0},\boldsymbol{T},\boldsymbol{T_M}) = \boldsymbol{D}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{T})\int_{\boldsymbol{Z}^*}^{\infty}\left(\boldsymbol{1}-\boldsymbol{D}(\boldsymbol{0},\boldsymbol{T},\boldsymbol{T_M})-\boldsymbol{K}\sum_{t_i>T}^{T_M}\boldsymbol{D}(\boldsymbol{0},\boldsymbol{T},\boldsymbol{t_i})\boldsymbol{h}\right)\boldsymbol{\varphi}(\boldsymbol{Z})\boldsymbol{dZ}$$

where

- $\boldsymbol{\varphi}(\boldsymbol{Z})$ is the standard normal density function evaluated at Z

Substituting for the discount bond formulas, then simplifying, we obtain

$$\boldsymbol{P_{HW}(0,T,T_M) = D(0,0,T)\int_{Z^*}^{\infty}\varphi(Z)dZ - D(0,0,T_M)\int_{Z^*}^{\infty}\varphi(Z)e^{-\frac{1}{2}v^2(0,T,T_M)-v(0,T,T_M)Z}dZ}$$

$$\boldsymbol{- Kh\sum_{t_i>T}^{T_M}\frac{D(0,0,t_i)}{D(0,0,T)}\int_{Z^*}^{\infty}\varphi(Z)e^{-\frac{1}{2}v^2(0,T,t_i)-v(0,T,t_i)Z}\quad dZ}$$

Now, the first integral will reduce to $N(-Z^*)$. The second integral, after making the substitution $Z^{(1)} = -Z - v(0,T,T_M)$, will reduce to $N(-Z^* - v(0,T,T_M))$. The third set of integrals, after making the substitution $Z^{(1)} = -Z - v(0,T,t_i)$, will reduce to the form $N(-Z^* - v(0,T,t_i))$. Making these substitutions, and letting $v(0,T,T_M) = v_M$ and $v(0,T,t_i) = v_i$ the price of a European payer swaption will therefore be

$$\boldsymbol{P_{HW}(0,T,T_M) = L(D(0,0,T)N(-Z^*) - D(0,0,T_M)N(-Z^* - v_m) - K\sum_{t_i>T}^{T_M}D(0,0,t_i)h_iN(-Z^* - v_i))}$$

proving equation (15).

For a receiver swaption, the same argument is followed, except that **(A.10)** becomes

$$\boldsymbol{R_{HW}(0,T,T_M) = L\,D(0,0,T)E_0\left[\max(K\sum_{t_i>T}^{T_M}D(0,T,t_i)h - 1 + D(0,T,T_M),0)\right]}$$

because the buyer of the swaption will now be entering into a receiver IRS. From this, the condition for exercise will now be

$$\boldsymbol{1 - \lambda(T,T_M,Z) > K\sum_{t_i>T}^{T_M}\lambda(T,t_i,Z)h}$$

but the value of Z^* will still be the same, if all the characteristics of the receiver swaption is the same as that of the payer swaption. We will therefore integrate for all values of $Z < Z^*$. Following the same integration steps as above, we obtain

$$\boldsymbol{R_{HW\,Swaption}(0,T,T_M) = L(-D(0,0,T)N(Z^*) + D(0,0,T_M)N(Z^* + v_m) + K\sum_{t_i>T}^{T_M}D(0,0,t_i)h_iN(Z^* + v_i))}$$

proving equation (16).

Option to enter into tail swap

From equation (12), $P_{swaption} = \xi(0)\left[\frac{\max(D(T)_{tail} + S_{T,K}, 0)}{\xi(\mathrm{T})}\right]$, we obtain the integral in the same fashion as in the short proof for the European Swaption

$$\boldsymbol{P_{HW\ tail}(0, T, T_M)} \\ \boldsymbol{= L.D(0,0,T)\int_{Z^{**}}^{\infty}\left(\frac{D_{tail}(T)}{L} + 1 - D(0,T,T_M) - K\sum_{t_i>T}^{T_M} D(0,T,t_i)h\right)\varphi(Z)dZ} \qquad \textbf{(A.12)}$$

except that the boundary condition for exercise has now changed to equation (24).

For simplicity, we will divide the previous formula up, which is made possible by the linearity property of integrals

$$\boldsymbol{P_{HW\ tail}(0, T, T_M) = D(0,0,T)\int_{Z^{**}}^{\infty} D_{tail}(T)\varphi(Z)dZ} \\ \boldsymbol{+ L.D(0,0,T)\int_{Z^{**}}^{\infty}\left(1 - D(0,T,T_M) - K\sum_{t_i>T}^{T_M} D(0,T,t_i)h\right)\varphi(Z)dZ}$$

Therefore,

$$\boldsymbol{P_{HW\ tail}(0, T, T_M) = LD(0,0,T)\int_{Z^{**}}^{\infty} D(T)_{tail}\varphi(Z)dZ + P(0, T, T_M)^{*}_{HW}}$$

Where $\boldsymbol{P(0, T, T_M)^{*}_{HW}}$ is the formula for a payer swaption, given by equation (15), but with one key difference: the boundary of the integral, Z^{**}, will be different from the case of a normal swaption, since the boundary condition for exercise is now equation (24) instead of equation (20).

Since

$$\mathbf{D_{tail}(T)} = \boldsymbol{L\left(\frac{D(0,T,T_1)}{D(0,T_0,T_1)} - 1 - K(\mathrm{T} - \mathrm{T_0})D(0,T,T_1)\right)}$$

from equation (9), and

$$\boldsymbol{D(0,t,T) = \frac{D(0,0,T)}{D(0,0,t)}\exp\left(-\frac{1}{2}v^2(0,t,T) - v(0,t,T)Z\right)}$$

we substitute into equation (A.12) to obtain

$$\begin{aligned}
&\boldsymbol{P_{HW\,tail}(0,T,T_M)}\\
&= \boldsymbol{D(0,0,T)\int_{Z^{**}}^{\infty}\left(\frac{D(0,0,T_0)}{D(0,0,T)}\exp\left(-\frac{1}{2}(v^2(T,T_1)\right.\right.}\\
&\boldsymbol{\left.-\ v^2(\overline{T_1},T_1) - \ (v(T,T_1) - v(\overline{T_1},T_1))Z)\right) - 1}\\
&\boldsymbol{\left.-\ K(\mathrm{T}-\mathrm{T_0})\frac{D(0,0,T_1)}{D(0,0,T)}\exp\left(-\frac{1}{2}v^2(0,T,T_1) - v(0,T,T_1)Z\right)\right)\varphi(Z)dZ}\\
&\boldsymbol{+\ P(0,T,T_M)^*_{HW}}
\end{aligned}$$

$$\begin{aligned}
\rightarrow\ &\boldsymbol{P_{HW\,tail}(0,T,T_M)}\\
&= \boldsymbol{D(0,0,T_0)\int_{Z^{**}}^{\infty}\exp\left(-\frac{1}{2}(v^2(T,T_1)\right.}\\
&\boldsymbol{\left.-\ v^2(T_0,T_1) - \ (v(T,T_1) - v(T_0,T_1))Z)\right)\varphi(Z)dZ - D(0,0,T)\int_{Z^{*}}^{\infty}\varphi(Z)dZ}\\
&\boldsymbol{-\ K(\mathrm{T}-\mathrm{T_0})D(0,0,T_1)\int_{Z^{**}}^{\infty}\exp\left(-\frac{1}{2}v^2(0,T,T_1) - v(0,T,T_1)Z\right)\varphi(Z)dZ}\\
&\boldsymbol{+\ P(0,T,T_M)^*_{swaption}}
\end{aligned}$$

Letting $v = v(0,T,T_1)$ and $\bar{v} = v(0,T_0,T_1)$, the first integral is solved by making the substitution $\bar{Z} = Z + v - \bar{v}$, which reduces it to $N(-Z^* + \bar{v} - v)\exp(v\bar{v} - \bar{v}^2)$. The second integral reduces to $N(-Z^*)$. The third integral, after making the substitution $\bar{Z} = Z + v$, reduces to $N(-Z^* - v)$. Substituting these values for the integrals into the above formula for the tail swaption, we obtain equation (22):

$$\begin{aligned}
&\boldsymbol{P_{HW\,tail}(0,T,T_M)}\\
&\boldsymbol{= D(0,0,T_0)N(-Z^{**} + \bar{v} - v)\exp\left(v\bar{v} - \bar{v}^2\right) - D(0,0,T)N(-Z^{**})}\\
&\boldsymbol{- K(T-T_0)D(0,0,T_1)N(-Z^{**} - v) + \ P(0,T,T_M)^*_{swaption}}
\end{aligned}$$

For a an option on a receiver tail swap, we integrate for all values of Z in the interval $[-\infty,\ Z^*]$, instead of $[Z^*,\infty]$, and define $S(t) = \boldsymbol{D(0,T,T_M) - 1 + K\sum_{t_i>T}^{T_M} D(0,T,t_i)h}$**, which yields**

$$\begin{aligned}
&\boldsymbol{R_{HW\,tail}(0,T,T_M)}\\
&\boldsymbol{= -D(0,T_0)N(Z^* - \bar{v} + v)\exp\left(v\bar{v} - \bar{v}^2\right) + D(0,0,T)N(Z^*)}\\
&\boldsymbol{+ K(T-T_0)D(0,0,T_1)N(Z^* + v) + \ R(0,T,T_M)^*_{swaption}}
\end{aligned}$$

which is equation (23).

Adjusting for collateral

Consider a payer swaption with regular collateral posting. Let the collateral threshold amount be defined by H. Also, for simplicity, disregard the grace period and assume that collateral is posted on every date that EFE is calculated. Gibson (2005:6) suggests that EFE can be calculated by taking expectation of

$$E\left[\max\left(\bar{S}_{T,K} - \max\left[\bar{S}_{T,K} - H, 0\right], 0\right)\right]$$

then remembering that $\overline{S(t)} = S(t) + D$, we have rather

$$E\left[\max\left(S_{T,K} + D - \max\left[S_{T,K} + D - H, 0\right], 0\right]$$

Now to integrate this, with two max functions, it needs to be observed that if $S_{T,K} + D > H$, then the above expression will reduce to

$$E[H] = \int_{S_{T,K} + D > H} H \boldsymbol{\varphi}\left(S_{T,K}\right) dS_{T,K}$$

If $0 < S(t) + D < H$, then since the inner max function is now zero, we have

$$E\left[S_{T,K} + D\right] = \int_{0 < S_{T,K} + D < H} \left(S_{T,K} + D(T)_{tail}\right) \boldsymbol{\varphi}\left(S_{T,K}\right) dS_{T,K}$$

Finally, if $S_{T,K} + D(T)_{tail} < 0$, we have

$$E[0] = 0$$

Therefore, the entire function is non-zero as long as $S_{T,K} + D(T)_{tail} > 0$, so therefore the outer max function will be integrated on all values for Z where $S_{T,K} + D(T)_{tail} > 0$, and the inner max function integrated on all values where $S_{T,K} + D(T)_{tail} > H$, as follows:

$$E\left[\max\left(S_{T,K} + D - \max\left[S_{T,K} + D(T)_{tail} - H, 0\right], 0\right] = \int_{S_{T,K} + D > 0} \left(S_{T,K} + D(T)_{tail} - \max\left[S_{T,K} + D(T)_{tail} - H, 0\right]\right) \boldsymbol{\varphi}(\boldsymbol{Z}) dz$$

From the linearity property of integrals, the max function and $S(t) + D$ can be integrated separately, since they are summed over each other. The max function will be integrated for values of Z where $S(t) + D > H$.

$$E\left[\max\left(S_{T,K} + D(T)_{tail} - \max\left[S_{T,K} + D(T)_{tail} - H, 0\right], 0\right] = \int_{S_{T,K} + D > 0} (S_{T,K} + D(T)_{tail}) \varphi(Z) dz - \int_{S_{T,K} + D > H} \left[S_{T,K} + D(T)_{tail} - H\right] \varphi(Z)\, dz \quad \textbf{(47)}$$

The first integral is the price of a tail swaption, as described in the previous section. Focusing on the second integral, the boundary condition can be expanded as

$$\frac{\boldsymbol{D_{tail}(T)}}{\boldsymbol{L}} + \boldsymbol{1 - D(0,T,T_M) = K\sum_{t_i>T}^{T_M} D(0,T,t_i)\,h_i + \frac{H}{L}}$$

$$\rightarrow \frac{\boldsymbol{\lambda(T,T_1,Z^{**})}}{\boldsymbol{\lambda(T_0,T_1,Z^{**})}} - \boldsymbol{1 - K(T-T_0)\lambda(T,T_1,Z^{**}) + 1 - \lambda(T,T_M,Z^{**})}$$
$$= \boldsymbol{K\sum_{t_i>T}^{T_M} \lambda(T,t_i,Z^{**})\,h_i + \frac{H}{L}}$$

Therefore Z^{**} is the standard normal random variable that satisfies the above equality. Since the boundary conditions for the first and second integrals are different, the value of the standard normal random variable satisfies the two boundary conditions will also be different. Therefore let Z^{*} be the standard normal random variable that satisfies the first integral's boundary condition, i.e. such that $S(t) + D = 0$. The second integral will be integrated over all values of $Z > Z^{**}$. This, together with the linearity of integrals property, leads to the following two integrals, derived from the second integral:

$$\int_{Z^{**}}^{\infty} (S_{T,K} + D_{tail}(T))\varphi(Z)dz - \int_{Z^{**}}^{\infty} H\varphi(Z)dz$$

The first integral will yield equation (22), except that Z^{**} will used as parameter to the standard normal CDF, instead of Z^{*}. The second integral will yield

$$HN(-Z^{**})$$

Therefore the second integral in equation (47) will yield

$$\boldsymbol{P^{**}{}_{HW\,tail}(0,T,T_M) - HN(-Z^{**})}$$

Where $\boldsymbol{P^{**}{}_{HW\,tail}(0,T,T_M)}$ is equation (22), except that the value of Z that satisfies

$$\frac{\boldsymbol{\lambda(T,T_1,Z^{**})}}{\boldsymbol{\lambda(T_0,T_1,Z^{**})}} - \boldsymbol{1 - K(\mathrm{T}-\mathrm{T}_1)\lambda(T,T_1,Z^{**}) + 1 - \lambda(T,T_M,Z^{**}) = K\sum_{t_i>T}^{T_M} \lambda(T,t_i,Z^{**})\,h_i + \frac{H}{L}}$$

instead of equation (24) will be used in the formula, which is denoted here Z^{**}.

Combining the two integrals from equation (47), we obtain

$$\boldsymbol{P_{collateral}(0,T,T_M) = P_{HW\,tail}(0,T,T_M) - P^{**}{}_{HW\,tail}(0,T,T_M) + HN(-Z^{**})}$$

For a receiver swap, similar arguments will yield

$$\boldsymbol{R_{collateral}(0,T,T_M) = R_{HW\,tail}(0,T,T_M) - R^{**}{}_{HW\,tail}(0,T,T_M) + HN(Z^{**})}$$

Proof of the swaption pricing formula under Black's model

A short proof of Black's formula for swaptions will be given here. This is relevant since the rationale behind the proof will be used in the following proof. We start by assuming that swap rates are log-normally distributed with the following equation, given by Bateson (2011:303), describing the swap rate at time T

$$\boldsymbol{s_T = s_0 \exp\left(-\frac{\sigma^2 T}{2} - \sigma Z\sqrt{T}\right)} \qquad \textbf{(A.13)}$$

Since we are taking the expectation $E[\max(S_{T,K}, 0)]$, which can be translated to the following,

$$\boldsymbol{A(0)L\,E[\max(s_T - K, 0)]} \qquad \textbf{(A.14)}$$

where,

- $A(0)$ is the annuity factor, defined by Hull(2012:639) as $A(T) = \sum_{i=0}^{M-1}(T_{i+1} - T_i)D(T, T_i)$ and where $T_0 = T$
- L is the notional amount
- K is the strike rate

and the expectation will be an integral over all the values of Z (since s_T is the only stochastic factor in the equation, which in turn contains only Z as a stochastic factor) for which $s_T - K > 0$, or for which $s_0 \exp\left(-\frac{v^2}{2} - vZ\sqrt{T}\right) > K$. Since the left hand side of the previous inequality is a decreasing function of Z, the inequality will be satisfied for all values of Z smaller than Z^*, which is that value of Z such that $s_0 \exp\left(-\frac{v^2}{2} - vZ\sqrt{T}\right) = K$. Therefore we have

$$\boldsymbol{\ln(s_0) - \left(\frac{\sigma^2 T}{2} + vZ^*\right) = \ln(K) \;\rightarrow\; Z^* = d_2 = \frac{\ln\left(\frac{s_0}{K}\right) - \frac{\sigma^2 T}{2}}{v\sqrt{T}}}$$

Therefore the following integral will be calculated

$$\boldsymbol{A(T)L\,E[\max(s_T - K, 0) = A(T)L\int_{-\infty}^{d_2}(s_T - K)f(z)dz}$$

$$\boldsymbol{= A(T)L\left(s_0\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{d_2}\exp\left(-\frac{\sigma^2 T}{2} - vz\sqrt{t}\right)\exp\left(-\frac{z^2}{2}\right)dz - K\int_{-\infty}^{d_2} f(z)dz\right)}$$

$$\boldsymbol{= A(t)L\left(s_0\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{d_2}\exp\left(-\frac{1}{2}\left(z + \sigma\sqrt{T}\right)^2\right)dz - KN(d_2)\right)}$$

If $(z + \sigma\sqrt{T})$ is substituted by $\bar{z}$ (change of variables), the top integration limit will change to $d_2 + \sigma\sqrt{T} = d_1$. Since the substitution transformed the integral into a normal cumulative distribution function (CDF) with d_2 as parameter, the final formula for the payer swaption is now

$$\boldsymbol{P(0,T,T_M)_{Black} = A(t)L(s_0N(d_1) - KN(d_2))}$$

For a receiver swaption, the same rationale is followed, except that the expectation is now taken by integrating for all values of $Z > d_2$. This yields

$$\boldsymbol{R(0,T,T_M)_{Black} = A(t)L(KN(-d_1) - s_0N(-d_2))}$$

Adjusting for a tail swap

When using Black model to calculate EE, the adjustment will also have to be made for the assumption that the entire period's cash flows are lost, should default occur between two payment dates. The difference between the ordinary swap at time T, $\mathrm{S}(0,\mathrm{T},\mathrm{T_M})$ and the tail swap at time T, $\overline{\mathrm{S}}(0,\mathrm{T},\mathrm{T_M})$, will be denoted by D again, and has the same definition as in the Hull-White case (Equation (9))

$$D_{tail}(T) = \overline{S}(0,T,T_M) - S(0,T,T_M) = L(\frac{D(0,T,T_1)}{D(0,T_0,T_1)} - 1 - K(T-T_0)D(0,T,T_1))$$

The expectation to calculate the payer swaption value will hence change to

$$\boldsymbol{\frac{\xi(0)E[\max(S(T,T,T_M) + D(T)_{tail},0)]}{\xi(T)} = D(0,0,T)A(T)L\,E\left[max\left(s_t - K + \frac{D_{tail}(T)}{A(T)},0\right)\right]}$$

Since the boundary condition for exercise has now changed to

$$\boldsymbol{s_T - K + \frac{D}{A(T)} = s_0\exp\left(-\frac{\sigma^2}{2}T - \sigma Z\sqrt{T}\right) - K + \frac{D(T)_{tail}}{L.A(T)} = 0} \qquad \textbf{(A.15)}$$

and the terms $\frac{D(\mathrm{T},\mathrm{T_1})}{D(\mathrm{T_0},\mathrm{T_1})} - 1$ in D_{tail}, taken together, represents a forward rate between times $\overline{T_1}$ and T, denoted $(T - T_0)f(0,T_0,T)$, which we will shorten to f_T. Since forward rates are log-normally distributed under Black's model, and is given by $(T - T_0)f_T = (T - T_0)f_0\exp\left(-\frac{v^2}{2}T - vZ\sqrt{T}\right)$. It will be assumed that the forward rate and forward swap rate have the same volatility v, which is an unrealistic, but necessary assumption, since this simplifies computation considerably. Since

$$\boldsymbol{\frac{D(T)_{tail}}{L} = f_0exp\left(-\frac{\sigma^2}{2}T - \sigma Z\sqrt{T}\right) - K(T-T_0)D(0,T,T_1)}$$

we can substitute into the boundary condition given in equation (A.15), and solving the boundary condition for Z, we get

$$Z = d_2 = \frac{\ln\left(\frac{\left(s_0 - \frac{f_0}{A(T)}\right)}{\left(K + \frac{K(T-T_0)D(0,T,T_1)}{A(T)}\right)}\right) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}} \qquad \textbf{(A.16)}$$

Now, in the same fashion as in the previous subsection, we will use this d_2 as the upper boundary in the integral used for expectation. The expression

$$A(0)LE\left[s_0 exp\left(-\frac{\sigma^2}{2}T - \sigma Z\sqrt{T}\right) - K + \frac{\left((T-T_0)f_0\exp\left(-\frac{\sigma^2}{2}\mathrm{T} - \sigma \mathrm{Z}\sqrt{\mathrm{T}}\right) - \mathrm{K}(T-T_0)\mathrm{D}(0,\mathrm{T},\mathrm{T_1})\right)}{A(T)}\right]$$

Is equal to

$$A(0)L\left(\left(s_0 + \frac{f_0(T-T_0)}{A(T)}\right)\int_{-\infty}^{d_1} exp\left(-\frac{\sigma^2}{2}T - \sigma z\sqrt{T}\right)f(z)dz - \left(K + \frac{\mathrm{K}(T-T_0)\mathrm{D}(0,\mathrm{T},\mathrm{T_1})}{A(T)}\right)\int_{-\infty}^{d_1} f(z)dz\right)$$

Evaluating the integrals, we obtain an expression for $P(0,T,T_M)_{Black\ tail}$

$$A(0)L\left(\left(s_0 + \frac{f_0(T-\bar{T})}{A(T)}\right)N(d_1) - \left(K + \frac{K(T-T_0)D(0,T,T_1)}{A(T)}\right)N(d_2)\right) \qquad \textbf{(A.17)}$$

which is for the payer swaption. For a receiver swaption, the same rationale is followed, except that the expectation is now taken for all values of $Z > d_2$. This yields

$$R(0,T,T_M)_{Black\ tail} = A(0)L\left(\left(K + \frac{K(T-T_0)D(0,T,T_1)}{A(T)}\right)N(-d_1) - \left(s_0 + \frac{f_0(T-T_0)}{A(T)}\right)N(-d_2)\right)$$

in the same fashion as in the previous subsection, with $d_1 = d_2 + \sigma\sqrt{t}$ (resulting from a change of variables that was necessary in the first integral).

Adjusting for collateral

Consider a payer swaption with regular collateral posting. Let the collateral threshold amount be defined by H. Also, for simplicity, disregard the grace period and assume that collateral is posted on every date that EFE is calculated. Gibson (2005:6) suggests that EFE can be calculated by taking expectation of

$$E\left[\max\left(\bar{S}_{T,K} - \max\left[\bar{S}_{T,K} - H, 0\right], 0\right)\right]$$

then remembering that $\bar{S}_{T,K} = S_{T,K} + D$, we have rather

$$E\left[\max\left(S_{T,K} + D_{tail}(T) - \max\left[S_{T,K} + D_{tail}(T) - H, 0\right], 0\right)\right]$$

Now to integrate this, with two max functions, it needs to be observed that if $S_{T,K} + D(T)_{tail} > H$, then the above expression will reduce to

$$E[H] = \int_{S_{T,K}+D>H} H dS_{T,K}$$

If $0 < S(t) + D < H$, then since the inner max function is now zero, we have

$$E\left[S_{T,K} + D_{tail}(T)\right] = \int_{0<S_{T,K}+D<H} \left(S_{T,K} + D_{tail}(T)\right) dS_{T,K}$$

Finally, if $S_{T,K} + D_{tail}(T) < 0$, we have

$$E[0] = 0$$

Therefore, the entire function is non-zero as long as $S_{T,K} + D_{tail}(T) > 0$, so therefore the outer max function will be integrated on all values for Z where $S_{T,K} + D_{tail}(T) > 0$, and the inner max function integrated on all values where $S_{T,K} + D_{tail}(T) > H$, as follows:

$$\begin{aligned} &E\left[\max\left(S_{T,K} + D(T)_{tail} - \max\left[S_{T,K} + D_{tail}(T) - H, 0\right], 0\right]\right. \\ &\quad = \int_{S_{T,K}+D(T)_{tail}>0} \left(S_{T,K} + D_{tail}(T) - \max\left[S_{T,K} + D_{tail}(T) - H, 0\right]\right) dz \end{aligned}$$

From the linearity property of integrals, the max function and $S(t) + D$ can be integrated separately, since they are summed over each other. The max function will be integrated for values of Z where $S_{T,K} + D(T)_{tail} > H$.

$$\begin{aligned} &E\left[\max\left(S_{T,K} + D(T)_{tail} - \max\left[S_{T,K} + D_{tail}(T) - H, 0\right], 0\right]\right. \\ &\quad = \int_{S_{T,K}+D>0} \left(S_{T,K} + D_{tail}(T)\right) dz - \int_{S_{T,K}+D>0} \max\left[S_{T,K} + D_{tail}(T) - H, 0\right] dz \\ &\quad = \int_{S_{T,K}+D>0} \left(S_{T,K} + D_{tail}(T)\right) dz - \int_{S_{T,K}+D>H} \left(S_{T,K} + D_{tail}(T) - H\right) dz \end{aligned}$$

Now, the boundary value for Z that satisfies $S_{T,K} + D_{tail}(T) > H$ is found in the same way as in the previous subsections, and is given by:

$$d_1 = \frac{\ln\left(\frac{\left(s_0 - \frac{f_0}{A(T)}\right)}{\left(\frac{H}{LA(T)} + K + \frac{K(T-T_0)D(0,T,T_1)}{A(T)}\right)}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{\left(s_0 - \frac{f_0}{A(T)}\right)}{\left(\frac{H}{LA(T)} + K + \frac{K(T-T_0)D(0,T,T_1)}{A(t)}\right)}\right) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$$

The first integral is just an option on a tail swap, so the EE under a (very simplified) collateral agreement for a payer swap, denoted $\boldsymbol{P_{BS\,Coll}(0,T,T_M)}$, is given by

$$\boldsymbol{P_{tail}} - \boldsymbol{A(t)L}\left(\left(\boldsymbol{s_0} + \frac{\boldsymbol{f_0(T-T_0)}}{\boldsymbol{A(t)}}\right)\boldsymbol{N(d_1)} - \left(\frac{\boldsymbol{H}}{\boldsymbol{LA(t)}} + \boldsymbol{K} + \frac{\boldsymbol{K(T-T_0)D(t,t_1)}}{\boldsymbol{A(t)}}\right)\boldsymbol{N(d_2)}\right)$$

Credit Valuation Adjustment

A proof is given here for equation (7), the formula to calculate CVA. The proof was adapted from a paper by Stein and Lee (2010:4-6). In the proof given by Stein and Lee, bonds and swaps were considered, however, for the sake of relevance, only swaps will be considered in this proof.

Let A and B be two counterparties. Consider a contract that A enters into with B, with price process $S(t,T)$ at time t when not adjusting for default risk. If B was to default at the current time (time t = 0), and the contract had positive value ($S(t,T) > 0$) to A, A then loses the amount $(1-R) \times \bar{S}(t,T)$, where R is the recovery rate and $\bar{S}(t,T)$ the price of a tail swap at time t. If the contract had negative value ($S(t,T) < 0$), A loses nothing. The current exposure to default is thus

$$(1-Rec) \times \max(\bar{S}(0,T),0)$$

If τ is the default time, the exposure to default at time τ is

$$(1-Rec) \times \max\,(\bar{S}(\tau,T),0) \tag{A.18}$$

Refer section The swaption approach to calculating CVA for an explanation of (A.18) and the usage of tail swaps. Now note that $\max\,(S(t,T),0)$ is the payoff of an option to buy or sell a swap contract at time t, that matures at time T. Taking the expectation of (A.18) with respect to a numeraire that has value $\xi(t)$ at time t, the following is obtained

$$\xi(0)E\left[\frac{(1-Rec)\max(\bar{S}(\tau,T),0)}{\xi(\tau)}1_{\tau<T}\right]$$

$$= (1-Rec)\xi(0)E\left[\int_0^T \frac{\max(\bar{S}(\tau,T),0)}{\xi(\tau)}\delta(t-\tau)dt\right]$$

$$= (1-Rec)\xi(0)\left[\int_0^T E\left[\frac{\max(\bar{S}(\tau,T),0)}{\xi(\tau)}\delta(t-\tau)\right]dt\right]$$

where $1_{\tau<T}$ is the indicator function of $(\tau < T)$ and δ is the Dirac delta function. Assume that default time is independent of contract value and the numeraire under the equivalent martingale measure. The expectation of the products is therefore equal to the product of the expectations, so then the above formula reduces to

$$(1-Rec)\xi(0)\left[\int_0^T E\left[\frac{\max(\bar{S}(\tau,T),0)}{\xi(\tau)}\right]E[\delta(t-\tau)]dt\right]$$

$$=(1-Rec)\left[\int_0^T \xi(0)E\left[\frac{\max(\bar{S}(\tau,T),0)}{\xi(\tau)}\right]E[\delta(t-\tau)]dt\right]$$

The first expectation in the integrand above is the price of an option to enter into a tail swap at time τ, while the second expectation is the density function of the distribution of default times, p, evaluated at τ, therefore $p(\tau)$. The formula for CVA therefore becomes

$$(1-Rec)\int_0^T \bar{S}(\tau,T)p(\tau)dt$$

proving equation (7).

Difference between a swap and a tail swap

A proof is given here for equation (9), the expression that describes the difference between a swap and a tail swap. The proof is adapted from Stein and Lee (2010:13-14). Stein and Lee allowed for different periodicities for the fixed and floating legs. In this paper, the simplifying assumption was made that fixed and floating legs have the same periodicity.

The time t price of a swap (paying fixed rate) can be represented as

$$S(t,T)=\sum_{i=0}^{n-1} f(t,t_i,t_{i+1})D(t,t_{i+1})h_i - C\sum_{i=0}^{n-1} D(t,t_{i+1})h_i \qquad \textbf{(A.19)}$$

where

- $n=(T-t)/h$
- $t_0=t$ is the first reset date

Since

$$f(t,t_i,t_{i+1})=\left(\frac{1}{h}\right)\left(\frac{D(t,t_i)}{D(t,t_{i+1})}-1\right)$$

Substituting the above into (A.19) yields

$$S(t,T)=D(t,t_0)-D(t,t_n)-C\sum_{i=0}^{n-1} D(t,t_{i+1})h_i \qquad \textbf{(A.20)}$$

Now, let $\bar{t}$, $\bar{t}_0$, etc. represent the same information for a tail swap. Refer to section 1.3 for a definition of tail swaps. All of the information is the same, except that the first reset date, $\bar{t}_0$, is before or on the valuation date, $\bar{t}$, i.e. $\bar{t}_0 < \bar{t}$. The first payment can therefore be represented at time t as

$$\frac{D(t,t_1)}{D(\bar{t}_0,t_1)} - D(t,t_1)$$

The value of the tail swap is then

$$\bar{S}(t,T) = \frac{D(t,t_1)}{D(\bar{t}_0,t_1)} - D(t,t_1) + \sum_{i=1}^{n-1} f(t,t_i,t_{i+1})D(t,t_{i+1})h_i - Ch_0D(t,t_0) - C\sum_{i=1}^{n-1} D(t,t_{i+1})h_i$$

$$\bar{S}(t,T) = \frac{D(t,t_1)}{D(\bar{t}_0,t_1)} - D(t,t_n) - C\bar{h}_0D(t,t_1) - C\sum_{i=1}^{n-1} D(t,t_{i+1})h_i \qquad \textbf{(A.21)}$$

To find the difference between the ordinary swap and tail swap, the difference between (A.20) and (A.21) is taken

$$\bar{S}(t,T) - S(t,T) = \frac{D(t,t_1)}{D(\bar{t_1},t_1)} - D(t,t_0) - C(\bar{h}_0 - h_0)D(t,t_1)$$

Since $t_0 = t$ and $\bar{h}_0$ corresponds to the full period, letting h denote the full period we obtain

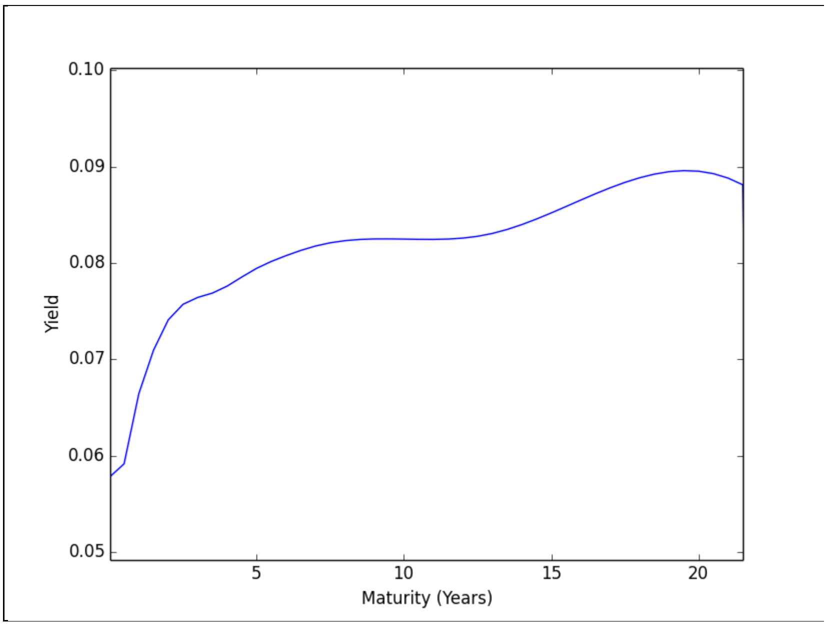
$$\bar{S}(t,T) - S(t,T) = \frac{D(t,t_1)}{D(\bar{t_1},t_1)} - 1 - C.(h - h_0)D(t,t_1)$$

Appendix B – Data Used

This section contains the data used for the calibration and eventual calculation of EE profiles and CVA for a South African scenario. Implied volatility matrices and term structure plots are listed under each date.

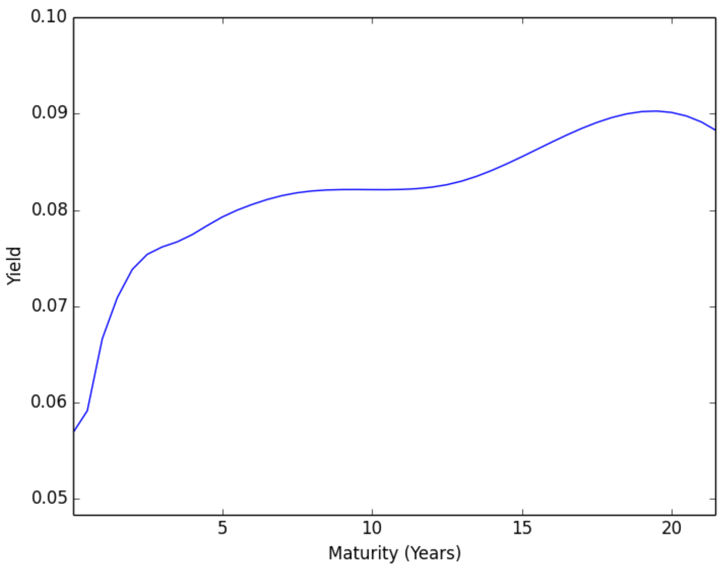
20/02/2014

		Tenors						
		1	2	3	4	5	7	10
Maturities	0.083333	0.164	0.217	0.205	0.182	0.151	0.17	0.197
	0.25	0.182	0.22	0.216	0.215	0.216	0.224	0.2
	0.5	0.198	0.219	0.222	0.207	0.203	0.199	0.196
	1	0.239	0.228	0.233	0.236	0.24	0.222	0.2
	2	0.23	0.236	0.222	0.211	0.204	0.198	0.191
	3	0.216	0.208	0.204	0.198	0.194	0.175	0.184
	4	0.191	0.189	0.187	0.184	0.183	0.18	0.176
	5	0.181	0.178	0.177	0.177	0.176	0.172	0.168
	7	0.164	0.163	0.162	0.16	0.159	0.155	0.156
	10	0.149	0.148	0.145	0.142	0.14	0.141	0.143



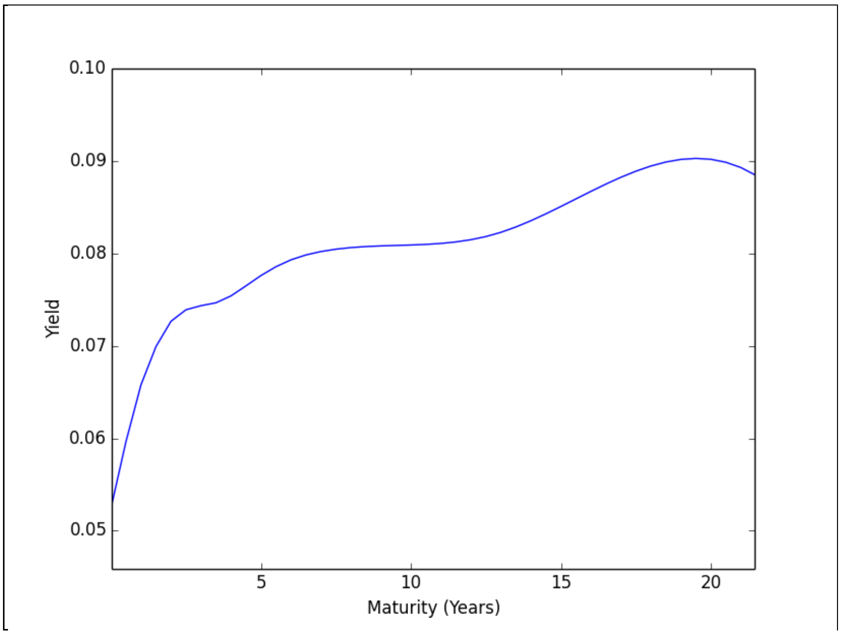
21/02/2014

		Tenors						
		1	2	3	4	5	7	10
Maturities	0.083333	0.164	0.218	0.207	0.183	0.152	0.171	0.198
	0.25	0.182	0.221	0.218	0.217	0.217	0.225	0.201
	0.5	0.199	0.221	0.224	0.209	0.205	0.2	0.197
	1	0.241	0.231	0.236	0.238	0.242	0.224	0.201
	2	0.234	0.238	0.224	0.212	0.204	0.2	0.192
	3	0.217	0.21	0.205	0.198	0.195	0.176	0.185
	4	0.193	0.19	0.186	0.185	0.184	0.18	0.177
	5	0.181	0.176	0.177	0.177	0.177	0.172	0.169
	7	0.171	0.167	0.165	0.162	0.161	0.157	0.159
	10	0.148	0.147	0.145	0.143	0.142	0.142	0.145



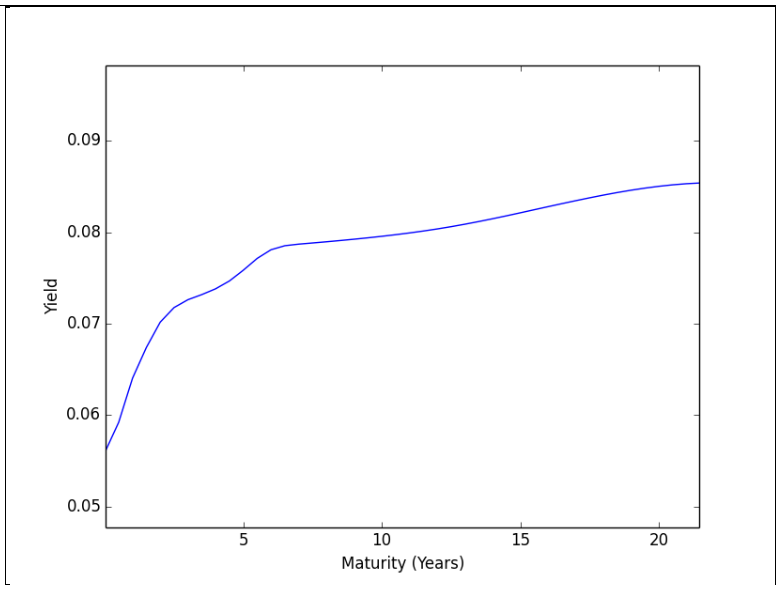
27/02/2014

		Tenors						
		1	2	3	4	5	7	10
Maturities	0.083333	0.166	0.22	0.209	0.185	0.153	0.172	0.2
	0.25	0.187	0.226	0.222	0.221	0.22	0.228	0.203
	0.5	0.198	0.221	0.222	0.202	0.194	0.194	0.199
	1	0.225	0.219	0.227	0.23	0.236	0.221	0.202
	2	0.235	0.242	0.225	0.214	0.206	0.2	0.194
	3	0.222	0.21	0.206	0.199	0.196	0.177	0.187
	4	0.189	0.19	0.186	0.184	0.183	0.181	0.178
	5	0.186	0.179	0.178	0.177	0.178	0.174	0.17
	7	0.166	0.164	0.165	0.163	0.162	0.0158	0.159
	10	0.151	0.149	0.147	0.145	0.143	0.143	0.145



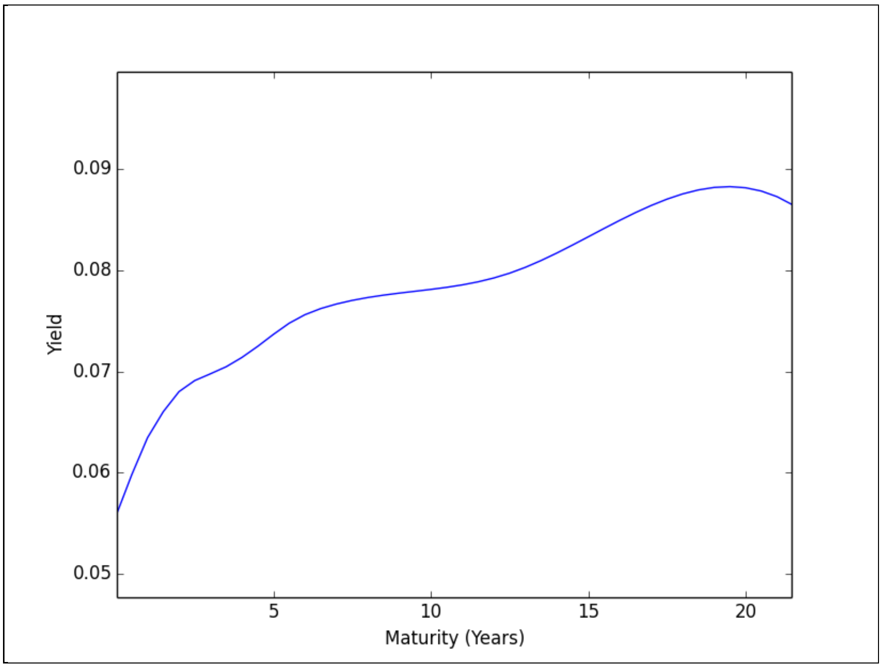
26/03/2014

		Tenors						
		1	2	3	4	5	7	10
Maturities	0.083333	0.172	0.229	0.222	0.202	0.173	0.176	0.181
	0.25	0.192	0.231	0.227	0.225	0.225	0.233	0.208
	0.5	0.2	0.223	0.223	0.206	0.2	0.201	0.204
	1	0.217	0.212	0.221	0.227	0.235	0.223	0.208
	2	0.243	0.247	0.231	0.219	0.212	0.206	0.198
	3	0.224	0.216	0.211	0.204	0.201	0.181	0.191
	4	0.198	0.196	0.192	0.19	0.189	0.185	0.184
	5	0.187	0.183	0.183	0.182	0.182	0.177	0.176
	7	0.171	0.169	0.167	0.166	0.165	0.163	0.164
	10	0.157	0.155	0.154	0.152	0.151	0.149	0.15



19/06/2014

		Tenors						
		1	2	3	4	5	7	10
Maturities	0.083333	0.081	0.113	0.126	0.131	0.127	0.133	0.141
	0.25	0.138	0.171	0.167	0.164	0.163	0.172	0.157
	0.5	0.157	0.174	0.179	0.173	0.174	0.169	0.165
	1	0.195	0.186	0.187	0.187	0.188	0.183	0.179
	2	0.247	0.248	0.226	0.21	0.199	0.19	0.177
	3	0.226	0.218	0.21	0.202	0.199	0.179	0.188
	4	0.201	0.198	0.195	0.193	0.191	0.19	0.188
	5	0.188	0.186	0.187	0.186	0.185	0.185	0.184
	7	0.178	0.174	0.173	0.171	0.171	0.17	0.173
	10	0.159	0.157	0.157	0.157	0.157	0.159	0.162



Appendix C – Python Code

This section contains Python code for selected algorithms. The actual code used to obtain the results is far more extensive than what is listed here. Including everything would require a whole document on its own.

The function to calculate EE on a specific date under Hull-White model follows. The function returns two values: The first is the value of the swaption/EE on a date *expDate.* The second returns the DV01 measure of the first return value. The *tail* parameter was used to determine whether we are pricing $\boldsymbol{P_{HW}(0,T,T)}$ or $\boldsymbol{P_{HW\ tail}(0,T,T_M)}$**.** Under no collateral agreements, *threshold* was zero. Under a collateral agreement, *threshold* was zero for the $\boldsymbol{P_{HW\ tail}(0,T,T_M)}$ portion of the formula, and equal to the collateral threshold for the $\boldsymbol{P^{**}_{HW\ tail}(0,T,T_M) + HN(-Z^{**})}$ portion of the formula. The rest of the parameters are self-explanatory. Note this function is spread over the next 4 pages.

```
#prices a swaption or EE with the given arguments

  def priceSwaption(self, expDate, strike, principal, endDate = -1,
                    tail = True, n = 1, payer = True, threshold = 0):

    if endDate - expDate < 0.31:
      printout = 1;
    else:
      printout = 0;
    sub = False
    annuity = self.Annuity_f(0, expDate, endDate, printout);

    if (compFloat(expDate, endDate) >= 0):
      print("Swaption expiry date must be before swap end date...returing zero")
      return [0, 0]

    #set default argument for endDate
    if (endDate == -1):
      endDate = self.dates[len(self.dates)-1]

    if (not(self.isRegular(endDate))):
      sub = self.subTermStruct(endDate);

    toNext = self.distToDates(expDate)[1]   #how long until the next payment date,
                                             #from the current expiry date
    if (not(compFloat(toNext, self.dt) == 0)):   #check if expDate does not fall
                                                            #on one of the regular payment dates.
      posDel = self.modTermStruct(expDate)     #if it does, this function adds
                                                                    #the new date and interpolates
                                                                    #(or extrapolates) the new value
                                                                    #on termStruct
      t1bar = self.dates[posDel-1]
      t = self.dates[posDel]
      t1 = self.dates[posDel+1]
      h1 = toNext
      regular = False
    else:
      posDel = self.getDateIndex(expDate)
      t = self.dates[posDel]
      t1 = self.dates[posDel+1]
      t1bar = t
      h1 = toNext
      regular = True
```

```
if (expDate > 0):

        if payer:
           z = self.solve(margin=0.0000001, expDate=expDate,
                       strike=strike + (threshold/principal/annuity),
                       posDel=posDel, endDate=endDate, tail=tail,
                       regular = regular)       #solve for z, to get the
                                                                         #boundary value
        else:
           strikeArg = strike - (threshold/principal/annuity)
           if strikeArg <= 0:
              strikeArg = 1e-10;
           z = self.solve(margin=0.0000001, expDate=expDate,
                       strike=strikeArg, posDel=posDel, endDate=endDate,
                       tail=tail, regular = regular)   #solve for z, to get
                                                                                  #the boundary value

     else:
        z = 0

     ti = expDate + toNext;
     expPos = posDel;

     count = 0
     sumPmt = 0
     sumPmtD = 0
     for j in self.dates[(expPos + 1):self.getDateIndex(endDate) + 1]:

        if (count == 0):
           incr = toNext;
        else:
           incr = self.dt;
        if payer:
           sumPmt = sumPmt
                 + self.D(0, 0, j)*norm.cdf(-z
                                   - sqrt(self.vsq(0, expDate, j)))*incr;

           sumPmtD = sumPmtD
                  + self.DV01_ZC(j)*norm.cdf(-z
                                    - sqrt(self.vsq(0, expDate, j)))*incr
        else:
           sumPmt = sumPmt
                 + self.D(0, 0, j)*norm.cdf(z
                                   + sqrt(self.vsq(0, expDate, j)))*incr;

           sumPmtD = sumPmtD
                  + self.DV01_ZC(j)*norm.cdf(z
                                    + sqrt(self.vsq(0, expDate, j)))*incr;

        count = count + 1
```

```
if payer:      #if we are pricing a payer swaption

        if (compFloat(expDate, 0) == 0):
            optionVal = principal*max(n*self.fwdSwap(0, 0, endDate)
            - strike, 0)*sumPmt;  #If the expiry date is today,
                                                            #returns the expected value of the
                                                            #swaption with constants, since all
                                                      #values are known

            if optionVal == 0:
                delta = 0
            else:
                subSwap = n*self.fwdSwap(0, 0, endDate)
                tempTermStruct = self.termStruct
                self.termStruct = asarray(self.termStruct)
                self.termStruct = self.termStruct + 0.0001
                delta = self.fwdSwap(0, 0, endDate) - subSwap
                self.termStruct = tempTermStruct
            return [optionVal, delta]

        v = exp(-sqrt(self.vsq(0, t, t1)*self.vsq(0, t1bar, t1))
        + self.vsq(0, t1bar, t1))
        cdf = norm.cdf(-z + sqrt(self.vsq(0, t1bar, t1))
        - sqrt(self.vsq(0, t, t1)))

        if (tail):
            D = self.D(0, 0, t1bar)*v*cdf - self.D(0, 0, t)*norm.cdf(-z)
            - strike*(self.dt - h1)*self.D(0, 0, t1)*norm.cdf(-z
                        - sqrt(self.vsq(0, t, t1)))
            Ddelta = self.DV01_ZC(t1bar)*v*cdf - self.DV01_ZC(t)*norm.cdf(-z)
            - strike*(self.dt - h1)*self.DV01_ZC(t1)*norm.cdf(-z
                        - sqrt(self.vsq(0, t, t1)))
            if compFloat(D, 0) == 0:
                D = 0
                Ddelta = 0
        else:
            D = 0
            Ddelta = 0

        optionVal = principal*(D + n*self.D(0, 0, expDate)*norm.cdf(-z)
        - n*self.D(0, 0, endDate)*norm.cdf(-z
                    - sqrt(self.vsq(0, expDate, endDate)))
        - strike*sumPmt - (norm.cdf(-z)*threshold/principal))
        delta = principal*(Ddelta + n*self.DV01_ZC(expDate)*norm.cdf(-z)
        - n*self.DV01_ZC(endDate)*norm.cdf(-z
                    - sqrt(self.vsq(0, expDate, endDate)))
        - strike*sumPmtD)


        if (not(compFloat(modFloat(expDate, self.dt), 0) == 0)):
            del self.termStruct[posDel], self.dates[posDel];
        if (sub):
            self.restoreTermStruct()

        return [optionVal, delta];
```

```
else:       #if we are pricing a receiver swaption

        if (compFloat(expDate, 0) == 0):
            optionVal = principal*max(-n*self.fwdSwap(0, 0, endDate) +
                        strike, 0)*sumPmt;    #If the expiry date is today, returns the
                                                  #expected value of the swaption
                                                  #with constants, since all values are known

            if optionVal == 0:
                delta = 0;
            else:
                subSwap = n*self.fwdSwap(0, 0, endDate)
                tempTermStruct = self.termStruct
                self.termStruct = asarray(self.termStruct)
                self.termStruct = self.termStruct + 0.0001
                delta = -self.fwdSwap(0, 0, endDate) + subSwap
                self.termStruct = tempTermStruct
            return [optionVal, delta]

        v = exp(-sqrt(self.vsq(0, t, t1)*self.vsq(0, t1bar, t1))
                          +  self.vsq(0, t1bar, t1))


       cdf = norm.cdf(z - sqrt(self.vsq(0, t1bar, t1))
                          + sqrt(self.vsq(0, t, t1)))
        if (tail):
            D = -self.D(0, 0, t1bar)*v*cdf + self.D(0, 0, t)*norm.cdf(z)
                + strike*(self.dt - h1)*self.D(0, 0, t1)*norm.cdf(z
                           + sqrt(self.vsq(0, t, t1)))

            Ddelta = -self.DV01_ZC(t1bar)*v*cdf + self.DV01_ZC(t)*norm.cdf(z)
                   + strike*(self.dt - h1)*self.DV01_ZC(t1)*norm.cdf(z
                                + sqrt(self.vsq(0, t, t1)))

            if compFloat(D, 0) == 0:
                D = 0
                Ddelta = 0
        else:
            D = 0
            Ddelta = 0

        optionVal = principal*(D + -n*self.D(0, 0, expDate)*norm.cdf(z)
                  + n*self.D(0, 0, endDate)*norm.cdf(z
                                + sqrt(self.vsq(0, expDate, endDate)))
                  + strike*sumPmt - (norm.cdf(z)*threshold/principal))
        delta = principal*(Ddelta + -n*self.DV01_ZC(expDate)*norm.cdf(z)
                + n*self.DV01_ZC(endDate)*norm.cdf(z
                           + sqrt(self.vsq(0, expDate, endDate)))
                + strike*sumPmtD);
        if (not(compFloat(modFloat(expDate, self.dt), 0) == 0)):
            del self.termStruct[posDel], self.dates[posDel];
        if (sub):
            restoreTermStruct()

        return [optionVal, delta];
```

The function used to solve for Z^*, Z^{**}, etc. in Hull-White swaption/EE valuation is directly below. After that, the function used to calculate the DV01 derivative for each zero-coupon bond follows.

```
#function to solve for z*, as required for Hull-White swaption valuation

  def solve(self, margin, expDate, strike, posDel, endDate, tail = True,
        n = 1, regular = False):

     toNext = self.distToDates(expDate)[1]
     if (not(regular)):      #check if expDate does not fall on
                 #one of the regular payment dates
        t1bar = self.dates[posDel-1]
        t = self.dates[posDel]
        t1 = self.dates[posDel+1]
        h1 = toNext
     else:
        t = self.dates[posDel]
        t1 = self.dates[posDel+1]
        t1bar = t
        h1 = self.dt

     D = self.lambd(t1, 0.5, t)/self.lambd(t1, 0.5, t1bar)

     def func(z):
        if (self.getDateIndex(expDate) != -1):
           expPos = self.getDateIndex(expDate);
        if (tail):
           D = self.lambd(t1, z, t)/self.lambd(t1, z, t1bar)
           - 1 - strike*(self.dt - h1)*self.lambd(t1, z, t);
           if (compFloat(D, 0) == 0):
              D = 0;
        else:
           D = 0
        lhs = n*(D + 1 - self.lambd(endDate, z, expDate))
        rhs = 0
        count = expDate + toNext;
        for j in self.dates[(expPos + 1):(self.getDateIndex(endDate) + 1)]:
           if (rhs == 0):
              incr = toNext;
           else:
              incr = self.dt;
           rhs = rhs + self.lambd(j, z, expDate)*incr;
           count = count + self.dt
        rhs = rhs*strike
        return (lhs - rhs)**2

     z = scipy.optimize.minimize_scalar(func).x
     return z
```

```
def DV01_ZC(self, maturity, tol = 10**-8):

     if compFloat(maturity, 0) == 0:
        return 0

     newRate = -log(self.D(0, 0, maturity))/maturity + tol
     diff = exp(-newRate*maturity) - self.D(0, 0, maturity)

     return (diff/tol)*(10**(-4));
```

The following function was used to price a swaption, or calculate EE, under Black's model. The *tailSwap* and *collateral* parameters have the same meaning as the *tail* and *threshold* parameters in the Hull-White case.

```
def BS_Swaption(self, sigma, expDate, tenor, freq, strike = 0.05, fwdSwap = 0.05,                principal =
1, isPayer = True, tailSwap = True, collateral = 0):

    if compFloat(tenor, 0) == 0:
        return [0, 0]

    if expDate <= 0:
        expDate = 1e-10

    sumPmt = [0];
    annuity = round(self.Annuity_f(0, expDate, expDate + tenor), 9)
    if compFloat(annuity, 0) == 0:
        annuity = annuity + 1e-9
    if not(isPayer):
        collateral = -collateral

    if tailSwap:
        toNext = self.distToDates(expDate)[1]
        fwdSwap = fwdSwap + (1/self.expectedDisc(0, expDate - self.dt
        + toNext, expDate) - 1)/annuity

        strike = strike + strike*(self.dt - toNext)*self.expectedDisc(0,
        expDate, expDate + toNext)/annuity +
                collateral/annuity/principal
    else:
        strike = strike + collateral/annuity/principal

    if (strike <= 0):
        strike = 1e-10


    d_1 = d1(fwdPrice=fwdSwap, strike=strike, sigma=sigma,
            expDate=expDate, rfr=0)
    d_2 = d2(fwdSwap, strike, sigma, expDate)


    if compFloat(fwdSwap, 0) == 0:
        fwdSwap = fwdSwap + 1e-9

    if isPayer:
        return [principal*annuity*(fwdSwap*norm.cdf(d_1)
            - strike*norm.cdf(d_2)),
                        0.0001*principal*annuity*norm.cdf(d_1)]
    else:
        return [principal*annuity*(strike*norm.cdf(-d_2)
            - fwdSwap*norm.cdf(-d_1)),
                        0.0001*principal*annuity*(norm.cdf(d_1) - 1)]
```

The functions used to calibrate the model follows. The first function, *calibrate_sigma,* finds a value for σ, holding a constant. The second function finds a value for a, holding σ constant. The first function is therefore basically method 1 from section 2.4.6. Method 2 would comprise running the second function, *calibrate_a,* first, then *calibrate_sigma,* using the *a* from the first function. The third function, *calibrate,* comprises method 3.

```
def calibrate_sigma(self, matrix, maturities, tenors):
    def func(sigma):
        tempSigma = self.sigma
        self.sigma = sigma

        vec = [0]
        tenor_idx = 0
        mat_idx = 0
        for i in maturities:
            tenor_idx = 0
            for j in tenors[:len(tenors)]:
                IV = matrix[mat_idx][tenor_idx]
                PVmod = self.priceSwaption(i, self.fwdSwap(0, i, j),
                                1, j + i, False)[0]
                PV = self.BS_Swaption(IV, i, j, 1/self.dt, self.fwdSwap(0,
                                i, j), self.fwdSwap(0, i, j))[0]
                vec.append(PVmod/PV - 1)
                tenor_idx = tenor_idx + 1
            mat_idx = mat_idx + 1

        self.sigma = tempSigma
        return vec
        return sum(vec)
    temp_sigma = self.sigma
    ret = scipy.optimize.leastsq(func, temp_sigma);
    return ret
```

```
def calibrate_a(self, matrix, maturities, tenors):
    def func(a):
        vec = [0]
        tenor_idx = 0
        mat_idx = 0
        for i in maturities:     #i will be the row number (maturity)
            tenor_idx = 0
            for j in tenors[:(len(tenors)-1)]:  #j is the column number

                tenor_1 = tenors[tenor_idx + 1]
                sqrtVswap_1 = (self.D(0, 0, i)
                         - self.D(0, 0, i + j))*self.B(i, i + tenor_1, a)
                sqrtVswap_2 = (self.D(0, 0, i)
                         - self.D(0, 0, i + tenor_1))*self.B(i, i + j, a)
                IV_1 = matrix[mat_idx][tenor_idx + 1]
                IV_2 = matrix[mat_idx][tenor_idx]
                sumTerm = (sqrtVswap_1/sqrtVswap_2 - IV_1/IV_2)
                vec.append(sumTerm)
                tenor_idx = tenor_idx + 1
            mat_idx = mat_idx + 1
        return vec
        return sum(vec)

    temp_a = self.a
    ret = scipy.optimize.leastsq(func, temp_a)
    return ret;
```

```
def calibrate(self, matrix, maturities, tenors, lifetime):

    def func(vecPm):
        tempA = self.a
        tempSigma = self.sigma
        self.a = vecPm[0]
        self.sigma = vecPm[1]

        principal = 1000
        vec = [0]
        tenor_idx = 0
        mat_idx = 0

        for i in maturities:
            tenor_idx = 0
            for j in tenors[:len(tenors)]:
                co_terminal = i + j;

                if (co_terminal > lifetime - lifetime/10*3)
                                and (co_terminal < lifetime + lifetime/10*3):
                    weight = 1
                else:
                    weight = 0

                IV = matrix[mat_idx][tenor_idx]
                PVmod = self.priceSwaption(i, self.fwdSwap(0, i, j),
                                           principal, j + i, False)[0]
                PV = self.BS_Swaption(IV, i, j, 1/self.dt,
                                      self.fwdSwap(0, i, j), self.fwdSwap(0, i, j),
                                      principal)[0]
                vec.append(weight*(PVmod/PV - 1))
                tenor_idx = tenor_idx + 1
            mat_idx = mat_idx + 1

        self.a = tempA
        self.sigma = tempSigma
        return vec
        return sum(vec)

    a_init = float(self.calibrate_a(matrix, maturities, tenors)[0])
    sigma_init = float(self.calibrate_sigma(matrix, maturities, tenors)[0])
    ret = scipy.optimize.leastsq(func, [a_init, sigma_init])
    self.a = ret[0][0]
    self.sigma = ret[0][1]
```

Integration algorithm described in section 4.4:

```
def integrate(self, x, y):

    area = [0]
    for i in range(len(x)-1):
        incr = x[i+1] - x[i]
        yVal = (y[i+1] + y[i])/2
        area.append(incr*yVal)
    return sum(area)
```

Appendix D – Guide to software used

A brief guide to using the (custom made) software to calculate the results given in section 5 is given in this appendix. The program does not have a name (yet), and is far from perfect, as some refinements are yet to be done on it.

The first screen will look like this. The white block in the middle with the heading "interest rate environments" will list all the already created interest rate environments. Each environment is described by the Hull-White one-factor model, so an a and σ parameter is listed for each environment. Before anything can be done in the program, a new environment has to be created, by clicking the "Create new environment" button at the top, and each one can also be deleted by selecting it and clicking "Delete" in the top right corner.

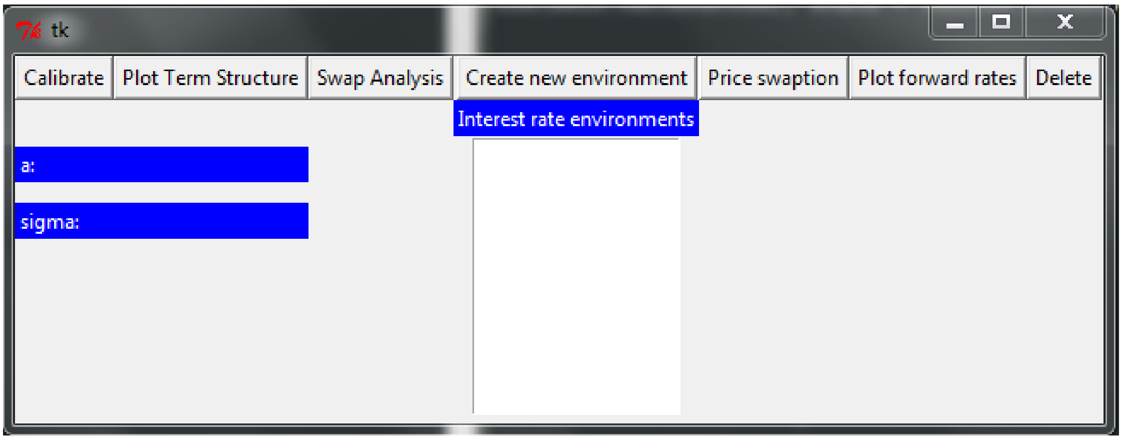


Figure 7-1: The opening screen of the software used in this paper

When "Create new environment" is clicked, the following screen will appear:

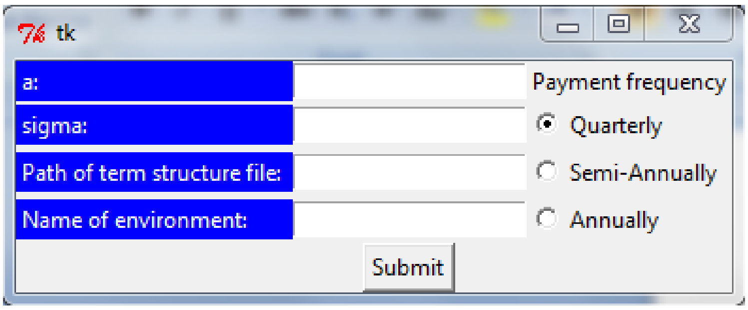


Figure 7-2: Details for each interest rate environment is entered here as it is created

In each corresponding field, a value for a and σ is entered, as well as the path of the file containing the term structure rates and dates, the name of the environment, and whether the payment frequency will be quarterly, semi-annually or annually for all swaps priced in the environment. In future versions of the program, environments will be able price swaps with different frequencies, but for the purposes of this paper, this sufficed. An example of how the term structure file **has to look**, is on the next page. The dates and maturity columns have to be in the columns that they are, but they can be any length. They have to be, of course, of the same length. The file also **has to be in .csv format.**

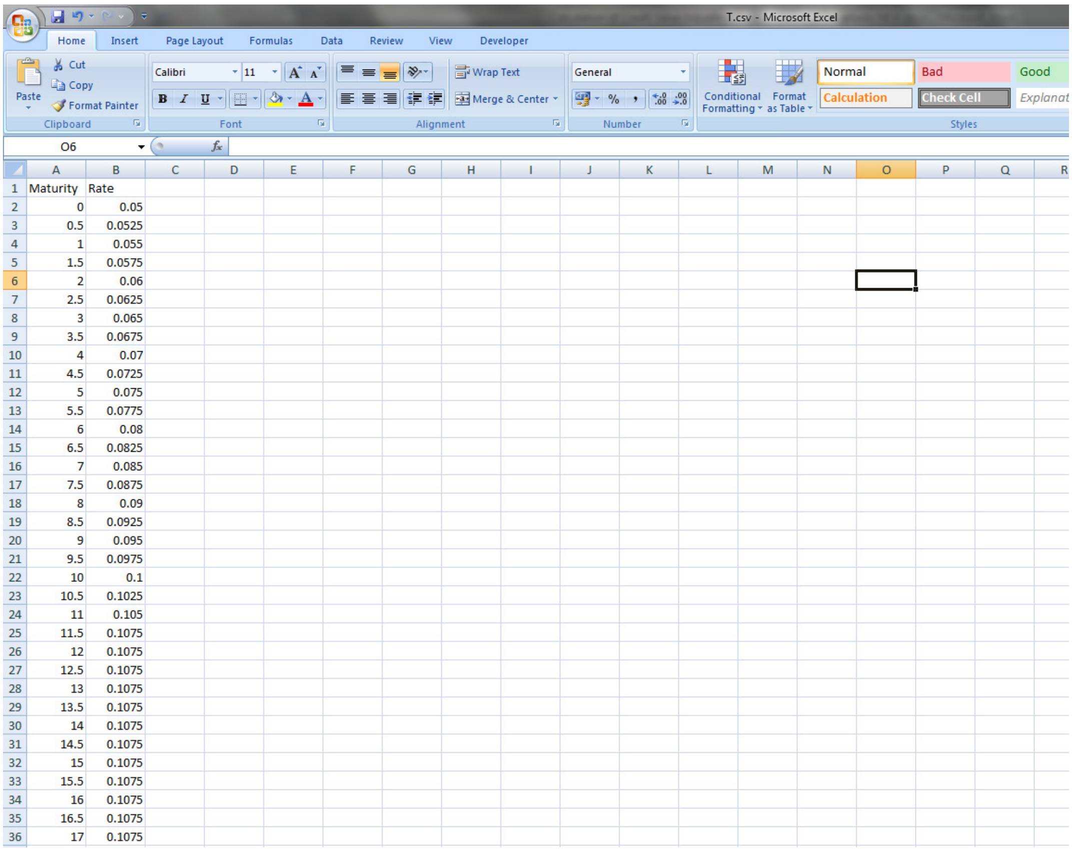


Figure 7-3: Format for the term structure (zero curve) input

After clicking "Submit" on the screen indicated by Figure 7-2, the main screen will look like this:

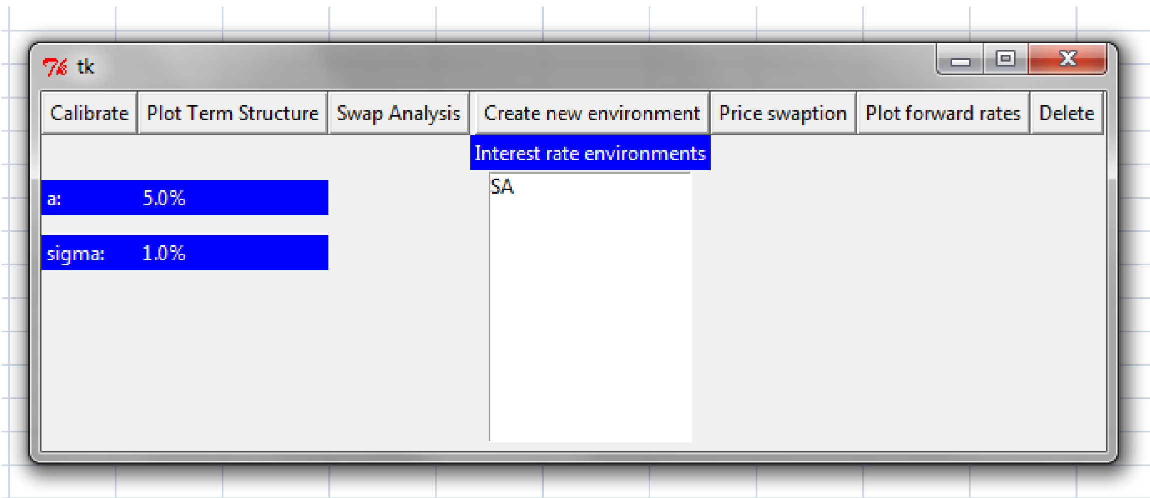


Figure 7-4: Main screen after an environment has been added

When an implied volatility matrix is available, the user will probably want to calibrate the interest rate model. The "Calibrate" button in the top left corner is for this purpose. A small menu will pop up when it is clicked, with the path of the .csv file containing the matrix (as with the term structure) and the remaining lifetime of the swap (in years) that is to be analysed. Please note that calibration may take a few minutes. Also note that if the implied volatility matrix is to be used when using

Black's model, calibration (unnecessarily) has to be done on the Hull-White model first. This is another feature that will be improved in the future. The format of the .csv file containing the implied volatility matrix also has a compulsory format, which looks as follows:

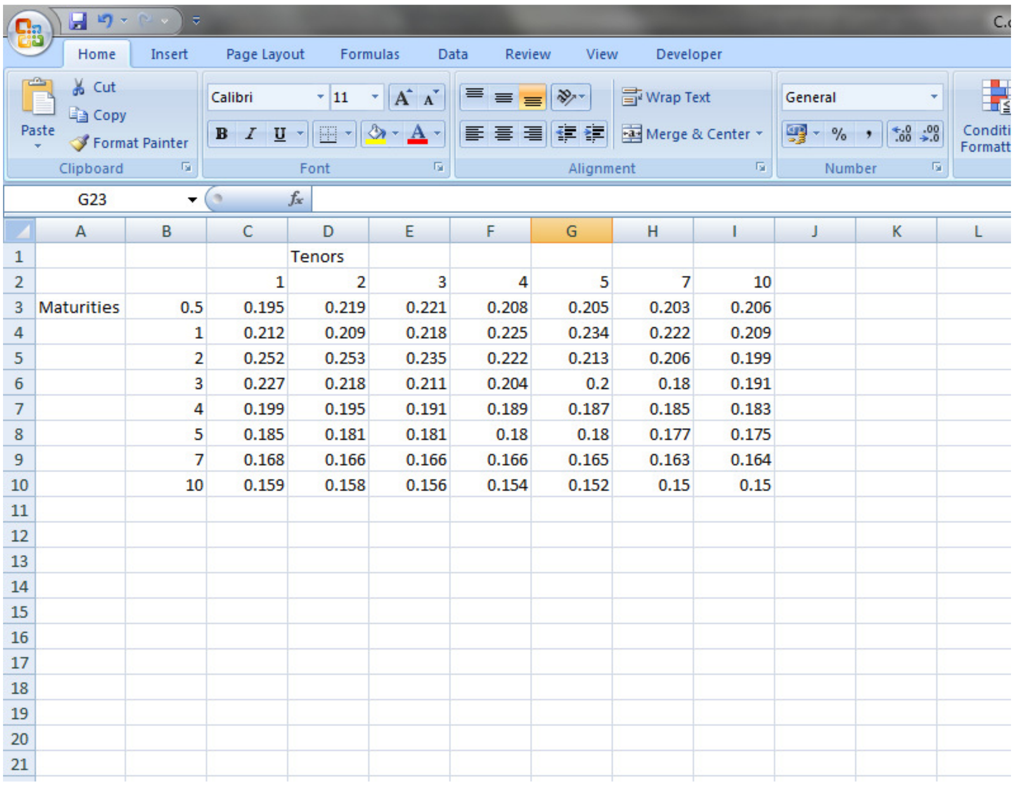


Figure 7-5: Format of the implied volatility matrix input

Any number of rows and columns may be added, though.

After an interest rate environment has been added, the "Swap Analysis" button will take the user to the following screen

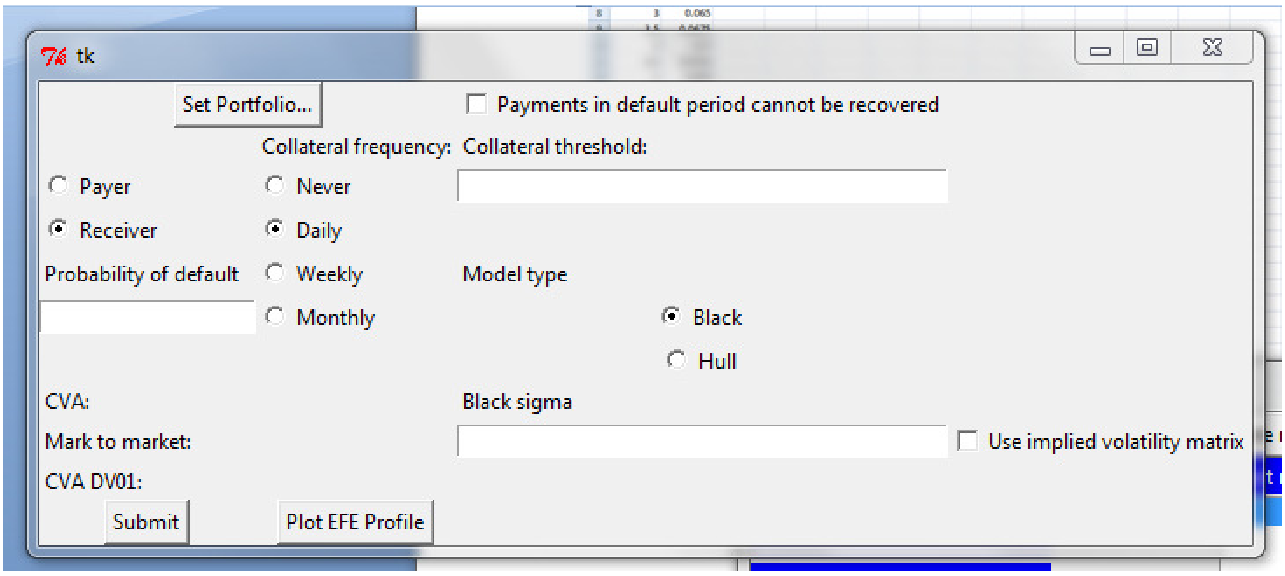


Figure 7-6: Screen where conditions of the swap to be analysed can be set

All the fields above are compulsory. When the user clicks the “Set Portfolio…” button, the following window will pop up:

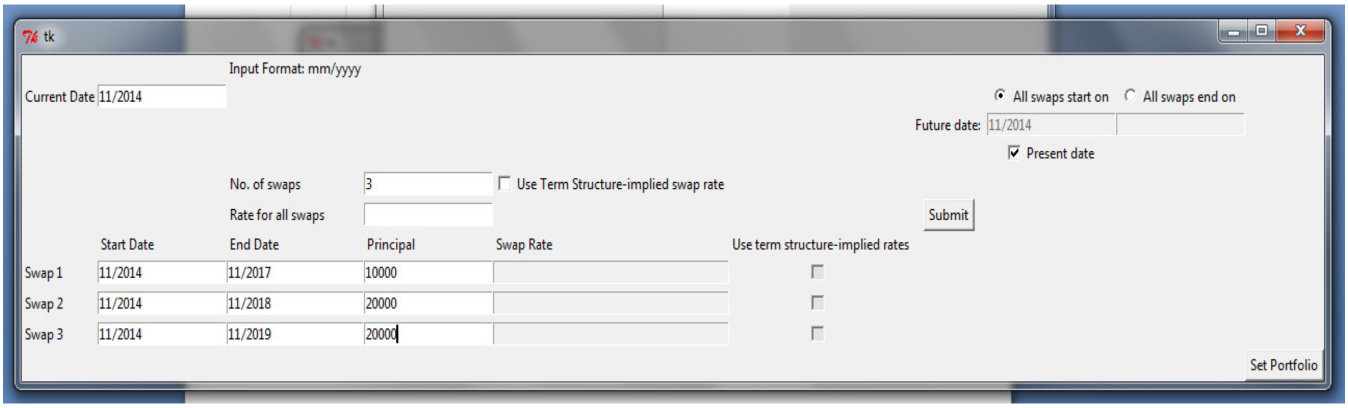


Figure 7-7: Window where swap portfolio is set up

Once again, all the fields are compulsory. When the window first pops up, everything below the “No. of swaps” label and field will not be there. It is only after setting how many swaps there should be, that the fields below will appear. In the fields at the top right, the starting or end dates for all the swaps can be set. It is highly recommended that all swaps either end or start on the same date, and that at least one swap starts on the present date. All swaps are also either payer or receiver swaps, and the swap rate is the same for all. This is to set up one of the scenarios described in section 2.1.2.1. The analysis of more complex swap portfolios is subject to more research and is outside the scope of this paper.

When the user is satisfied with all inputs, he/she should click “Set Portfolio” in the bottom right-hand corner, and if all the inputs in the window from Figure 7-6 is satisfactory, the “Submit” button of that window in the bottom left-hand corner will calculate CVA and the EE profile. The “Plot EFE Profile” button will plot the EE profile. It is recommended to keep the swap portfolio window open while doing analysis, so that the portfolio can be easily changed. Each time it has been changed, simply click the “Set Portfolio” button again.

The program is exited by closing the main window that lists all the interest rate environments.

8 List of References

Bateson, R.D. 2011. *Financial Derivative Investments: An Introduction to Structured Products.* London: Imperial College Press.

Ferrara, P.G. and Nezzamoddini, S.A. 2013. 'Interest Rate Swaps – An Exposure Analysis' URL: https://www.soa.org/Files/Research/Projects/research-2013-interest-rate-swaps.pdf (Accessed: 2 July 2014)

Gibson, M.S. 2005. 'Measuring Counterparty Credit Exposure to a Margined Counterparty' URL: http://www.federalreserve.gov/pubs/feds/2005/200550/200550pap.pdf (Accessed: 10 October 2014)

Gregory, J. 2012. *Counterparty Credit Risk and Credit Value Adjustments.* Chichester: John Wiley and Sons Ltd.

Gurrieri, S., Nakabayashi, M. and Wong, T. 2009. 'Calibration Methods of Hull-White Model' URL: http://papers.ssrn.com/sol3/Delivery.cfm/SSRN_ID1618606_code1185765.pdf?abstractid=1514192&mirid=1 (Accessed: 5 July 2014)

Harper, D. s.a. 'Introduction to Counterparty Risk' in *Investopedia.* URL: http://www.investopedia.com/articles/optioninvestor/11/understanding-counterparty-risk.asp (Accessed: 16 October 2014)

Harvey, C.R. 2011. 'Mark to market accounting'
URL: http://www.nasdaq.com/investing/glossary/m/mark-to-market-accounting (Accessed: 21 October 2014)

Henrard, M. 2005. 'Semi-Explicit Delta and Gamma for European Swaptions in Hull-White Model' URL: http://128.118.178.162/eps/fin/papers/0411/0411036.pdf (Accessed: 28 June 2014)

Hull, J. 2012. *Options, Futures, and Other Derivatives.* Harlow: Pearson Education Limited.

International Swaps and Derivatives Association (ISDA), 2009. 'Independent Amounts' URL: http://www.isda.org/c_and_a/pdf/Independent-Amount-WhitePaper-Final-October-22-2009.pdf (Accessed: 16 August 2014)

International Valuation Standards Council, 2013. 'Credit and Debit Valuation Adjustments' URL: http://www.ivsc.org/sites/default/files/IVSC%20CVA%20-DVA%20%20ED_0.pdf (Accessed: 11 October 2014)

Johannes, M. and Sundaresan, S. 2002. 'The impact of collateral on swap rates' URL: http://finance.bi.no/~finance/seminar/Sundaresan.pdf (Accessed: 15 October 2014)

McGlinchey, R. 2014. 'IFRS 13 – Accounting for CVA and DVA' URL: http://www.globalcapital.com/article/mbcdtzzyqt34/ifrs-13-accounting-for-cva-amp-dva (Accessed: 11 October 2014)

Pykhtin, M. and Zhu, S. 2007. 'A Guide to Modelling Counterparty Credit Risk' URL: http://papers.ssrn.com/sol3/Delivery.cfm/SSRN_ID1086265_code180495.pdf?abstractid=1032522&mirid=1 (Accessed: 9 April 2014)

Quadrini, V. and Wright, R.E. 2012. 'Finance, Banking and Money (v. 1.1)'. URL: http://2012books.lardbucket.org/books/finance-banking-and-money-v1.1/index.html (Accessed: 13 October 2014)

Rand Merchant Bank. 2014. 'Fixed Income Rates'.
URL:
http://www.rmb.co.za/GlobalMarkets/rates_historicalRates.asp?product=Bonds&instrument=R186&rateDisplay=R186&firsttime=1 (Accessed: 19 October 2014)

Reis, R. and Watson, M.W. 2007. 'Measuring Changes in the Value of the Numeraire'. URL: http://www.economicdynamics.org/meetpapers/2007/paper_324.pdf> (Accessed: 13 October 2014)

Ro, S. 2014. 'Government Bond Yields Around the World are Making New Lows Day In and Day Out' in *Business Insider* URL: < http://www.businessinsider.com/global-10-year-government-bond-yields-2014-8 (Accessed: 2 September 2014)

Shearman and Sterling LLP. 2013. 'Basel III Framework: The Credit Valuation Adjustment (CVA) Charge for OTC Derivative Trades'.
URL:
http://www.shearman.com/en/newsinsights/publications/2013/11/basel-iii-framework (Accessed: 28 March 2014)

Sorensen, E.H. and Bollier, T.F. 1994. 'Pricing Swap Default Risk' URL:
http://www.jstor.org/stable/4479743?origin=JSTOR-pdf (Accessed: 21 February 2014)

Stein, H. and Lee, K.P. 2010. 'Counterparty Valuation Adjustments' URL: http://papers.ssrn.com/sol3/Delivery.cfm/SSRN_ID1785813_code732372.pdf?abstractid=1463042&mirid=1 (Accessed: 21 February 2014)

Urroz, G.E. 2001. 'Ordinary Differential Equations with Scilab'. URL: http://www.infoclearinghouse.com/files/scilab/scilab09.pdf (Accessed: 2 October 2014)

Van Dalen et al. 2014. 'Housing Market Recovery Continues'. URL: https://economics.rabobank.com/publications/2014/august/housing-market-recovery-continues/
(Accessed: 2 September 2014)